Lecture 27: Final Review

1)
$$\lim_{x\to 2} \frac{\frac{1}{x^2}-\frac{1}{4}}{\frac{1}{x}-\frac{1}{2}} = \lim_{x\to 2} \frac{\left(\frac{1}{x}-\frac{1}{2}\right)\left(\frac{1}{x}+\frac{1}{2}\right)}{\frac{1}{x}-\frac{1}{2}} = 1$$

2)
$$\frac{d}{dx}\left[\cos^2(\tan(x))\right] = f'(g(x))\cdot g'(x) = -\sin(2\tan x)\cdot \sec^2(x)$$

ⓐ $f(y) = \cos^2(y)$ $\quad g(x) = \tan(x)$

$f'(y) = -2\cos(y)\sin(y)$ $\quad g'(x) = \sec^2(x)$

$= \sin(2y)$

ⓑ Compute derivative to $f(x) = \ln(2x^{\sin x})$

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left[\ln 2 + (\sin x)\ln x\right]$$
$$= \sin x \cdot \frac{1}{x} + \cos x \cdot \ln x$$

3) A curve in the xy-plane is defined by the parametric equations

$$x(t) = 3t^2+1 \qquad y(t) = 2t^3+1$$

ⓐ At what point on the curve does the tangent line have slope 3?

$$3 = \frac{dy}{dx} = \frac{y'}{x'} = \frac{6t^2}{6t} = t$$

So the point is $(28, 55)$

~~ⓑ Find the equations of all tangent lines to the curve that pass through (5,1)~~

$$\cancel{\begin{cases} 3t^2+1 = 5 \\ 2t^3+1 = 1 \end{cases} \Leftrightarrow \begin{cases} t = \pm\frac{2}{\sqrt{3}} \\ \end{cases}}$$

(3) Find the equations of all tangent lines that pass through $(5,1)$

Tangent line at t is

$$y-(2t^3+1)=t(x-(3t^2+1))$$

plug in $(x,y)=(5,1)$

$$1-(2t^3+1)=t(5-(3t^2+1))$$

$$\Leftrightarrow \quad -2t^3=-3t^3+4t$$

$$\Rightarrow \quad t^3-4t=0$$

$$t(t^2-4)=0 \Leftrightarrow t(t-2)(t+2)=0$$

$$\Leftrightarrow t=0,2,-2.$$

So the lines are

$$y-1=0 \Leftrightarrow y=1$$

$$y-17=2(x-13) \Leftrightarrow y=2x-9$$

$$y-(-15)=-2(x-13) \Leftrightarrow y=-2x+11.$$

4) Suppose $y=f(x)$ is defined implicitely $\sin(x+y)+\cos(x+y)=e^{3x}$ and that $f(0)=0$

(a) Compute $f'(0)$

$$\frac{d}{dx}\left[\sin(x+y)+\cos(x+y)\right]=\frac{d}{dx}\left[e^{3x}\right]$$

$$\cos(x+y)(1+y')-\sin(x+y)(1+y')=3e^{3x}$$

$$(1+y')\cdot\left[\cos(x+y)-\sin(x+y)\right]=3e^{3x}$$

$$\Rightarrow y'=\frac{3e^{3x}}{\cos(x+y)-\sin(x+y)}-1$$

See D.

So $$f'(0)=\frac{3e^{0}}{\cos(0)-\sin(0)}-1=2$$

(b) Use the tangent line approximation to ~~approximate~~ $f(-0.1)$

Linearization:

$$L(x)=2(x-0)+f(0)=2x$$

$$\rightarrow f(-0.1)\approx L(-0.1)=-0.2$$

(c) $$f''(x)=\frac{d}{dx}\left[\frac{3e^{3x}}{\cos(x+y)-\sin(x+y)}-1\right]=$$

$$=\frac{(\cos(x+y)-\sin(x+y))\left[\frac{d}{dx}3e^{3x}\right]-\frac{d}{dx}\left[\cos(x+y)-\sin(x+y)\right]\cdot 3e^{3x}}{(\cos(x+y)-\sin(x+y))^2}$$

$$= \left(\cos(x+y) - \sin(x+y)\right) \cdot 9e^{3x}$$

$$\cancel{\left[-\sin(x+y)(1+y') - \cos(x+y)(1+y')\right] 3e^{3x}}$$

$$\cancel{\left(\cos(x+y) - \sin(x+y)\right)^2}$$

Plug in $(x,y) = (0,0)$

$y' = 2$

$$\Rightarrow f''(0) = \frac{9 - (-3) \cdot 3}{1^2} = 18$$

(c) Is the graph of f at $x=0$ concave up, down, or neither? Up

#5) Insert picture of $f'(x)$ from question 5 on Final Exam (Spring 2016)

(a) List all intervals where f is decreasing

$(-2,2)$, $(8,10)$

(b) List all intervals where f is concave up

$(-6,-4)$, $(0,1,3)$, $(3,5)$

(c) List the x coordinates of all critical numbers of f and identify them as local minimizers, local maximizers, or neither.

Critical numbers: -2, 2, 3, 8

local max, local min, local neither, local max

(d) What is

$$\lim_{x\to 8} \frac{f'(x)}{x-8} = \lim_{x\to 8} \frac{f'(x)-f'(8)}{x-8} =$$

$$= f''(8) = \frac{0-4}{8-5} = -\frac{4}{3}$$

(e) If $f(4)=0$, is $f(8)$ positive or negative?

positive

(f) If $f(1) = -1.25$, use the tangent line approximation to estimate $f(0.85)$

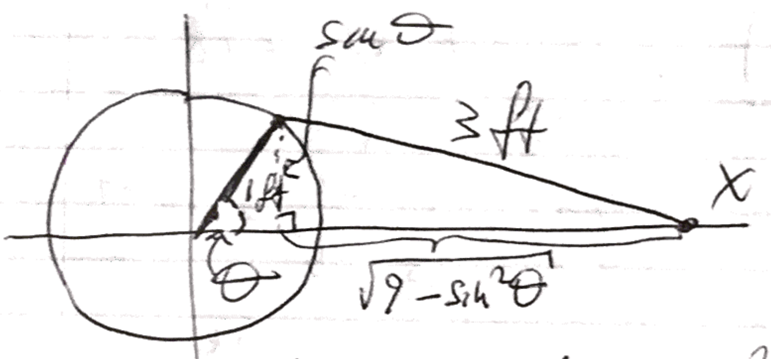
$$L(x) = f(1) + f'(1)(x-1)$$

$$= -1.25 - 3(x-1)$$

$$L(0.85) = -1.25 - 3(-.15) = -1.25 + .45 =$$

$$= -0.8$$

6) One end of a rigid rod of length 3 ft is attached to a wheel of radius 1 ft centered at the origin. The wheel is free to rotate about the origin. As the wheel rotates, the right end of the rod slides along the x-axis



ⓐ Find the equation relating x and θ

Height of triangle is $\sin\theta$.

So $x = \cos\theta + \sqrt{9-\sin^2\theta}$

ⓑ Suppose that when $\theta = \frac{\pi}{3}$, the right end of the rod is moving to the right at a rate of 0.5 ft/sec. How fast is θ changing when $\theta = \frac{\pi}{3}$?

$$\frac{d}{dt}X = \frac{d}{dt}\left[\cos\theta + \sqrt{9-\sin^2\theta}\right]$$

$$X' = -\sin\theta \cdot \theta' + \frac{1}{2\sqrt{9-\sin^2\theta}} \cdot (-2\sin\theta\cos\theta)\theta'$$

Plug in $X' = \frac{1}{2}$, $\theta = \frac{\pi}{3}$

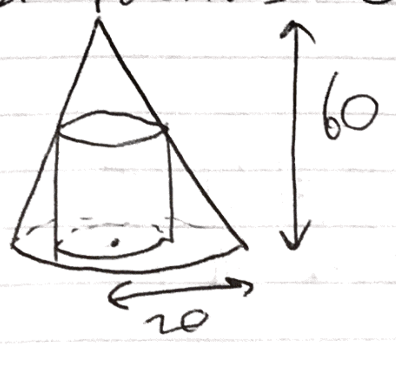
$$\Rightarrow \frac{1}{2} = -\frac{\sqrt{3}}{2} \cdot \theta' + \frac{1}{2\sqrt{9-\frac{3}{4}}} \cdot \left(-2 \cdot \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)\right)\theta'$$

$$= -\frac{\sqrt{3}}{2}\theta' + \frac{1}{\sqrt{33}} \cdot \left(-\frac{\sqrt{3}}{2}\right)\theta'$$

$$= \frac{\sqrt{3}}{2}\theta'\left(-1 - \frac{1}{\sqrt{33}}\right)$$

$$\Rightarrow \theta' = \frac{1}{-\sqrt{3}\left(1+\frac{1}{\sqrt{33}}\right)} = -\frac{1}{\sqrt{3}+\frac{1}{\sqrt{11}}}$$

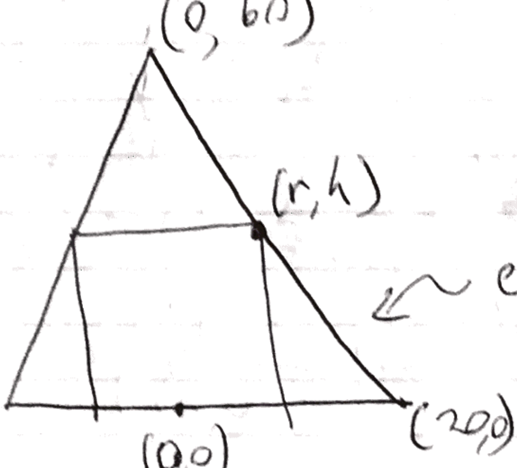
7) Find the height h and radius r of the cylinder of maximum volume that can be inscribed in a cone of radius 20 cm and height 60 cm.



Cross-section

Let V be the area of ~~rectangle~~ cylinder

$V = \pi r^2 h$



equation of line

$$y = 60 + \left(\frac{0-60}{20-0}\right)(x)$$
$$= 60 - 3x$$

So

max $V = \pi r^2 h$

subject to

$h = 60 - 3r$

$\Rightarrow$ max $V = \pi r^2 (60 - 3r)$

with

$20 \geq r \geq 0$

$$0 = V' = \frac{d}{dr}(60\pi r^2 - 3\pi r^3) = 120\pi r - 9\pi r^2$$
$$= \pi r(120 - 9r)$$

$\Leftrightarrow r = 0$ or $r = \frac{40}{3}$

endpoint

Candidates:

$r = 40/3$, $\underbrace{0, 20}_{\text{endpoints}}$

crit point

$V(\frac{40}{3}) = \pi\left(\frac{40}{3}\right)^2 \cdot 20 = \frac{20 \cdot 1600\pi}{3}$

$V(0) = 0$

$V(20) = 0$

So maximal r is $40/3$

with $h = 60 - 3r = 20$.

8) Consider the function $f(x) = \frac{x+1}{x} + 2\tan^{-1}(x)$

ⓐ Determine all horizontal and vertical assymptotes of $f(x)$.

Vertical: $\lim_{x \to 0^+} \frac{x+1}{x} + 2\tan^{-1}(x) = \lim_{x \to 0^+} 1 + \frac{1}{x} + \lim_{x \to 0^+} 2\tan^{-1}(x)$

$= +\infty$

$\lim_{x \to 0^-} \frac{x+1}{x} + 2\tan^{-1}(x) = -\infty$

So $x = 0$ is a vertical asymptote.

Horizontal: $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 1 + \frac{1}{x} + 2\tan^{-1}(x) = 1 + \pi$ ← horizontal

$\lim_{x \to -\infty} f(x) = 1 - \pi$ ← asymptotes

ⓑ Compute $f'(x)$

$f'(x) = -\frac{1}{x^2} + \frac{2}{1+x^2}$

ⓒ Find critical numbers

$0 = f'(x) \Leftrightarrow \frac{2}{1+x^2} = \frac{1}{x^2} \Leftrightarrow 2x^2 = 1 + x^2 \Leftrightarrow x = \pm 1$

$f(1) = 2 + \frac{\pi}{2}$

$f(-1) = -\frac{\pi}{2}$

(d) Determine all intervals on which f is increasing

Interval	f'
$(-\infty,-1)$	$+$
$(-1,0)$	$-$
$(0,1)$	$-$
$(1,\infty)$	$+$

$$f'(x) = -\frac{1}{x^2} + \frac{2}{1+x^2} = -\frac{(1+x^2)}{x^2(1+x^2)} + \frac{2x^2}{x^2(1+x^2)} =$$

$$= \frac{x^2-1}{x^2(1+x^2)}$$

Increasing on $(-\infty,-1)$, $(1,\infty)$

(e) Compute $f''(x)$

$$f''(x) = \frac{2}{x^3} - 2(1+x^2)^{-2} \cdot 2x = \frac{2}{x^3} - \frac{4x}{(1+x^2)^2}$$

(f) Give x-coordinates of inflection points.

$$0 = f''(x) \iff 0 = \frac{2(1+x^2)^2 - 4x^4}{x^3(1+x^2)^2} = \frac{2+4x^2-2x^4}{x^3(1+x^2)^2} =$$

$$= -2\frac{(x^4-2x^2-1)}{x^3(1+x^2)}$$

set $y = x^2$ and solve for y.

$$x^2 = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

So $x = \pm\sqrt{1+\sqrt{2}}$

(g) Sketch the graph of $f(x)$.