Lecture 26: Optimization

Continued from last time:

Ex: Sketch the graph of $f(x) = \frac{\sin(x)}{\cos(x)+1}$

① Domain: All $x \neq \pi k$ ← odd integer

② Intercepts:

$f(x) = 0 \iff \sin(x) = 0$ and $\cos(x) \neq -1$.

$\iff x = \pi k$ ← even integer

↗ x-intercepts

$f(0) = 0$ ← y-intercept

③ Asymptotes:

Vertical $\lim\limits_{x \to \pi k^{\mp}} f(x) \overset{\text{L'Hospital}}{=} \lim \frac{\cos(x)}{-\sin(x)}$

(k: odd integer)

So $\lim\limits_{x \to \pi k^-} f(x) = +\infty$ (k odd)

$\lim\limits_{x \to \pi k^+} f(x) = -\infty$ (k odd)

$\Rightarrow x = \pi k$ (k odd) are vertical asymptotes

Horizontal:

$\lim\limits_{x \to \pm\infty} \frac{\sin x}{\cos(x)+1}$ undefined

No horizontal asymptotes

4) Symmetry:

$f(x+2\pi) = f(x)$ So f is periodic with period 2π.

[So only need to sketch for example between $[-\pi, \pi]$]

5) Increasing / Decreasing

$$f'(x) = \frac{(\cos(x)+1)\cos x \quad \sin x(-\sin x)}{(\cos x+1)^2}$$

$$= \frac{1+\cos x}{(\cos x+1)^2} = \frac{1}{1+\cos x}$$

$\Rightarrow f'(x) = 0$?? never

no critical points

$f'(x) > 0$ always on domain

So increasing.

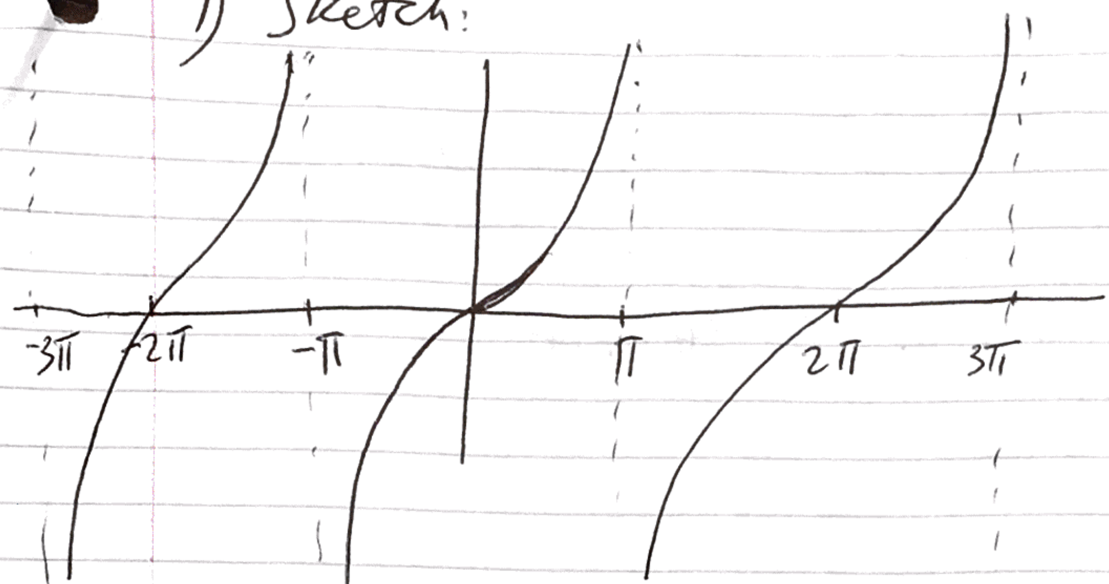
6) Concave up / Down:

$$f''(x) = -\frac{1}{(1+\cos x)} \cdot -\sin x = \frac{\sin x}{1+\cos x}$$

$\Rightarrow f''(x) > 0 \Leftrightarrow \sin x > 0.$

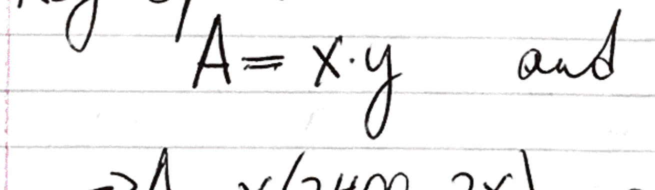
$f''(x) < 0 \Leftrightarrow \sin x < 0$

7) Sketch:



Optimization:

Ex: A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?



Notation: y = length horizontal

x = width

A = area of rectangle

Key equations

$$A = x \cdot y \quad \text{and} \quad 2400 = 2x + y$$

$$\Rightarrow y = 2400 - 2x$$

$$\Rightarrow A = x(2400 - 2x) = 2400x - 2x^2$$

So need to find maximum of A on the interval $0 \le x \le 1200$

Closed Interval Method:

$0 = A'(x) = 2400 - 4x$

$\Leftrightarrow x = 600$

Candidates: $x = 600$, $\overbrace{0, 1200}^{\text{endpoints}}$

$\Rightarrow A(600) = 720{,}000$ ← crit number

$A(0) = 0$

$A(1200) = 0$

So maximizer is $x =$

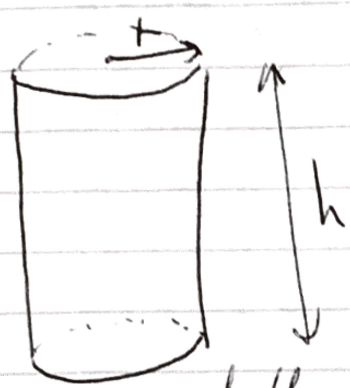
with $y = 2400 - 2(600) = 1200$

General Strategy:

1) Understand the problem: What are the variables? What is to be minimized/maximized? What are constraints on the variables?
2) Draw a picture
3) Introduce mathematical notation for relevant quantities
4) Express the function to be minimized/maximized in terms of a single unknown, by using constraints if necessary.
5) Use the methods we have learned to find the maximum or minimum.

Ex: A cylindrical can is to be made to hold 1000 cm^3 of oil. Find the dimensions that will minimize the amount of metal used to make the can?

A: surface Area
V: volume
h: height
r: base radius.



want this small

$$A = h \cdot (2\pi r) + 2(\pi r^2)$$
$$= 2\pi r h + 2\pi r^2$$

$$1000 = V = \pi r^2 h \Rightarrow h = \frac{1000}{\pi r^2}$$

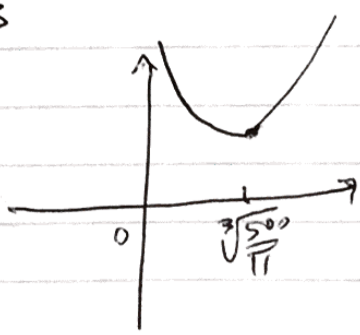
$$\Rightarrow A = \frac{2\pi r (1000)}{\pi r^2} + 2\pi r^2 = \frac{2000}{r} + 2\pi r^2$$

So we want to minimize $A(r)$ for $r > 0$.

$$0 = A'(r) = -\frac{2000}{r^2} + 4\pi r = \frac{-2000 + 4\pi r^3}{r^2}$$

$$\Leftrightarrow 4\pi r^3 = 2000 \Leftrightarrow r = \sqrt[3]{\frac{500}{\pi}}$$

critical number



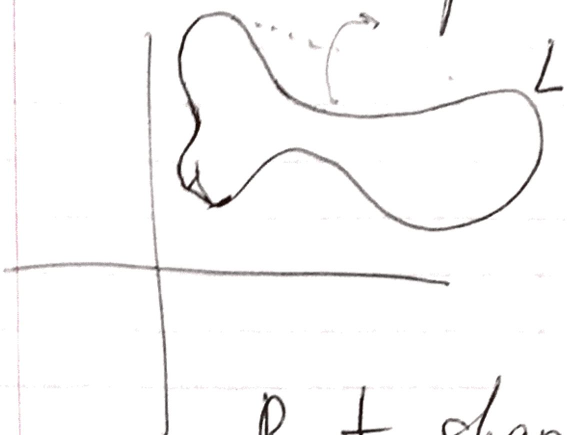
notice for $r > \sqrt[3]{\frac{500}{\pi}}$, $A'(r) > 0$ so increasing

So $r = \sqrt[3]{\frac{500}{\pi}}$ is the absolute minimum.

So optimal $r = \sqrt[3]{\frac{500}{\pi}}$

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}}$$

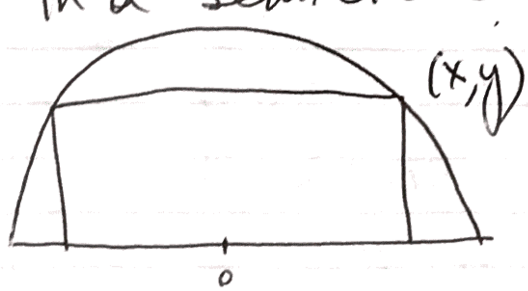
Aside: Suppose you would like to draw a curve in the plane of length L, that encloses maximal area. What shape should the curve be?



Best shape is a circle. (Isoperimetric Inequality)

Ex: Drop of water!

Ex: Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.



Let the semicircle be the upper half of the circle
$x^2+y^2=r^2$

Key Eqn:
$A=2xy$

Constraints
$x^2+y^2=r^2$
$r\geq y\geq 0.$
$r\geq x\geq 0$

$\Rightarrow\ y=\sqrt{r^2-x^2}$

So $A(x)=2x\sqrt{r^2-x^2}$

$A'(x)=2\sqrt{r^2-x^2}-\frac{2x^2}{\sqrt{r^2-x^2}}=\frac{2(r^2-2x^2)}{\sqrt{r^2-x^2}}$

product rule

So critical point is $x=\frac{r}{\sqrt{2}}$

Candidates: $x=\frac{r}{\sqrt{2}},\ 0,\ r$

Then $A(\frac{r}{\sqrt{2}})=r^2$
$A(0)=0$
$A(r)=0$

So the Area of largest inscribed rectangle is r^2.