Lecture 25: Curve Sketching

Guideline for sketching a curve $y = f(x)$

(A) Determine the domain of f

(B) If possible, determine the x-intercept and y-intercept.

(C) Symmetry:

① Even function $f(-x) = f(x)$

② Odd function $f(-x) = -f(x)$

③ Periodic $f(x+p) = f(x)$ for all x.

(D) Asymptotes:

① Horizontal asymptote: $L = \lim_{x \to \infty} f(x)$

$L = \lim_{x \to -\infty} f(x)$

② Vertical assymptotes:

$\lim_{x \to a^{\pm}} f(x) = \pm\infty$

(E) Intervals of Increase or Decrease:

Find intervals where $f' > 0$ and $f' < 0$

(F) Local maxima and minima: find critical numbers and use either the first derivative test or second derivative to test for local minimum / maximum

(G) Concavity: Compute $f''(x)$ use concavity test

(H) Sketch the curve

Ex: Sketch $y = \frac{2x^2}{x^2-1}$

A. Domain is $x \neq -1, 1$

B. x-intercept: $0 = \frac{2x^2}{x^2-1} \Leftrightarrow x = 0$

y-intercept: $\frac{2(0)^2}{0^2-1} = 0$

C. This function is even

D. $\lim_{x \to +\infty} \frac{2x^2}{x^2-1} = \lim_{x \to +\infty} \frac{2}{1 - 1/x^2} = 2$

$\lim_{x \to -\infty} \frac{2x^2}{x^2-1} = 2$.

Horizontal Asymptote $= y = 2$

$\lim_{x \to 1^+} \frac{2x^2}{x^2-1} = +\infty$

$\lim_{x \to 1^-} \frac{2x^2}{x^2-1} = -\infty$

$\lim_{x \to -1^-} \frac{2x^2}{x^2-1} = +\infty$

$\lim_{x \to -1^+} \frac{2x^2}{x^2-1} = -\infty$

$x = 1, -1$ are vertical asymptotes

E. $f'(x) = \frac{(x^2-1)\left[\frac{d}{dx} 2x^2\right] - 2x^2\left[\frac{d}{dx}(x^2-1)\right]}{(x^2-1)^2}$

$= \frac{-4x}{(x^2-1)^2}$

interval	f'
$(-\infty, 0)$	+
$(0, \infty)$	−

F unique critical point is $x=0$
$x=0$ is a local maximizer

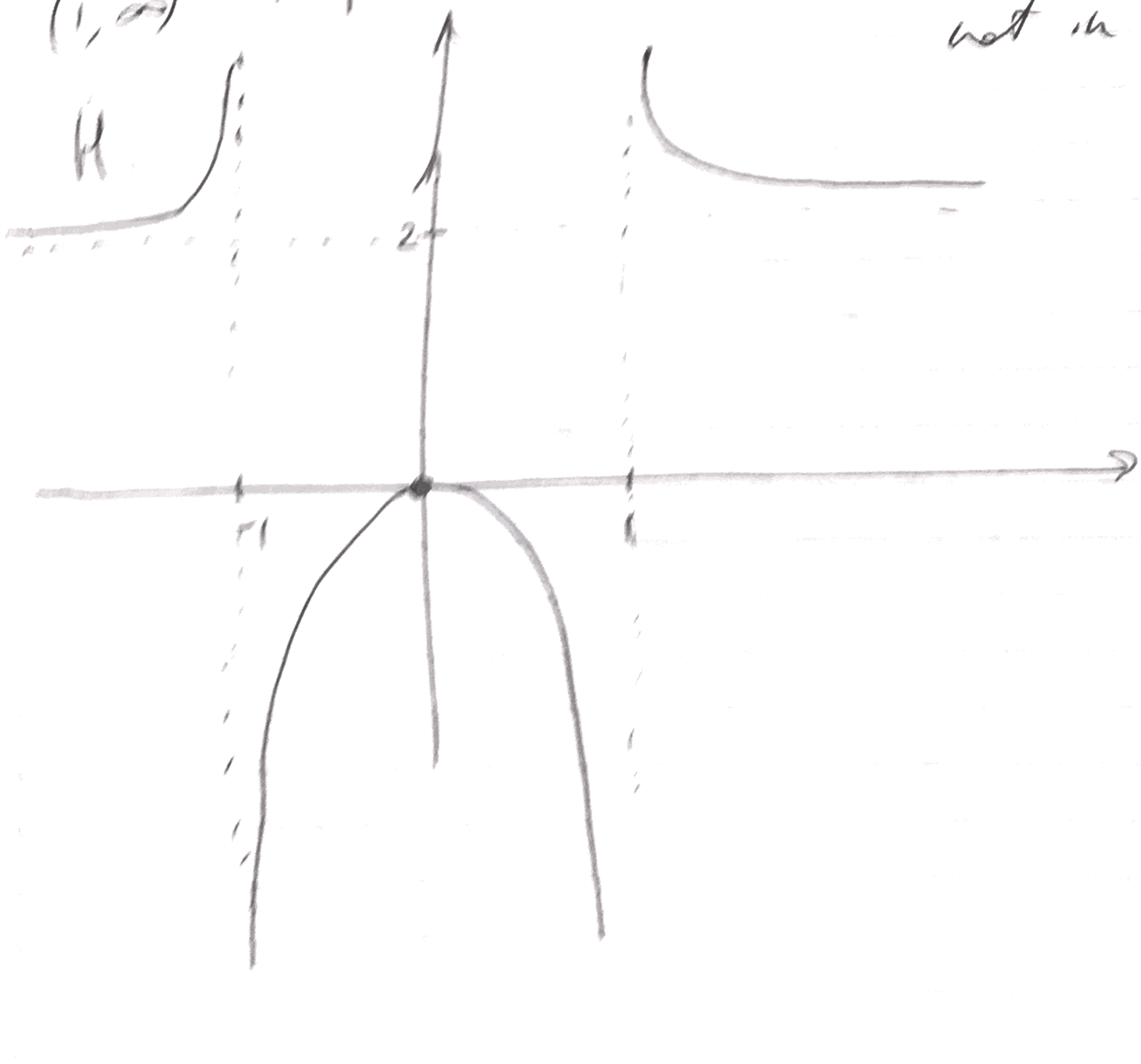
G $f''(x) = \frac{(x^2-1)^2 \frac{d}{dx}(-4x) - (-4x)\frac{d}{dx}(x^2-1)^2}{(x^2-1)^4} =$

$= \frac{12x^2+4}{(x^2-1)^3}$ $\qquad 12x^2+4 > 0$

interval	f''
$(-\infty,-1)$	+
$(-1,1)$	−
$(1,\infty)$	+

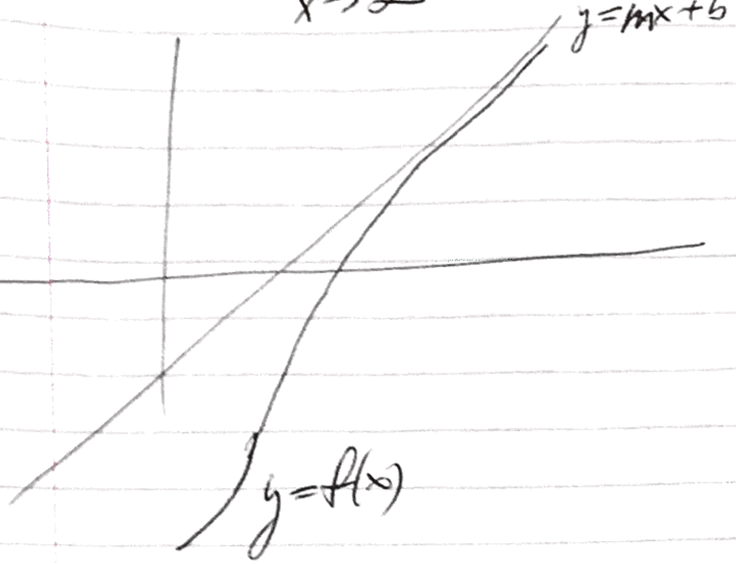
No inflection points because $-1, 1$ are not in domain

H



Defn: The line $y=mx+b$ is called a slant asymptote, if

$$\lim_{x\to\infty}\left[f(x)-(mx+b)\right]=0$$



Ex: Sketch the graph of $f(x)=\dfrac{x^3}{x^2+1}$

A. Domain in $(-\infty,\infty)$

B. x-intercept $y=0$, y-intercept $x=0$

C. Odd function $f(-x)=-f(x)$

D. There are no vertical asymptotes

$\lim_{x\to\infty} f(x)=\infty$, $\lim_{x\to-\infty} f(x)=-\infty$

No Horizontal asymptotes

But

$$\lim_{x\to\infty} f(x)-x=\lim_{x\to\infty}\frac{x^3}{x^2+1}-x=$$

$$=\lim_{x\to\infty}\frac{x^3-(x^3+x)}{x^2+1}=\lim_{x\to\infty}\frac{-x}{x^2+1}=0$$

So $y=x$ is a slant asymptote

E. $f'(x) = \dfrac{x^2(x^2+3)}{(x^2+1)^2}$

So $f' > 0$ always

F. $f'(0) = 0 \iff x = 0$

We cannot conclud whether
0 is a local minimizer
or maximizer
because f' does not change sign
and $f'' = 0$ here

G. $f''(x) = \dfrac{2x(3-x^2)}{(x^2+1)^3}$

$f'(x) = 0 \iff x=0, \; x = \pm\sqrt{3}$

Interval	f''
$(-\infty, -\sqrt{3})$	+
$(-\sqrt{3}, 0)$	−
$(0, \sqrt{3})$	+
$(\sqrt{3}, +\infty)$	−

H.