Lecture 24: L'Hospital's Rule

Ex: How to compute $\lim_{x \to 1} \frac{\ln x}{x-1}$?

Thm: [L'Hospital's Rule]

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a)

Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

$\boxed{\text{Indeterminate type of } \frac{0}{0}}$

or

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

$\boxed{\text{Indeterminate type of } \frac{\infty}{\infty}}$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right-hand-side exists (or is ∞ or $-\infty$)

Note: The rule is valid for one-sided limits or for limits $x \to \infty$ and $x \to -\infty$

Ex: Find $\lim_{x \to 1} \frac{\ln x}{x-1}$

Let's check assumptions

Set $f(x) = \ln x$

$g(x) = x - 1$

$f'(x) = \frac{1}{x}$

$g'(x) = 1$ [$\neq 0$ near $x=1$]

$$\begin{cases} \lim_{x \to 1} f(x) = 0 \\ \lim_{x \to 1} g(x) = 0 \end{cases}$$

So

L'Hospital: $\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$

Ex: Calculate $\lim_{x \to \infty} \frac{e^x}{x^2}$

$$\begin{cases} \lim_{x \to \infty} e^x = \infty \\ \lim_{x \to \infty} x^2 = \infty \end{cases}$$ and $\frac{d}{dx} x^2 = 2x$ is not zero for x large

$\overset{\text{L'Hospital}}{\Longrightarrow} \lim_{x \to \infty} \frac{e^x}{x^2} = \lim \frac{\frac{d}{dx} e^x}{\frac{d}{dx} x^2} = \lim_{x \to \infty} \frac{e^x}{2x}$

Set $f(x) = e^x$

$g(x) = 2x$

$$\lim_{x\to\infty} f(x) = \infty \quad , \quad \lim_{x\to\infty} g(x) = \infty .$$

and $\underset{\underset{2}{\parallel}}{g'(x)} \neq 0$ for x large.

L'Hospital

$$\Rightarrow \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = +\infty .$$

So

$$\lim_{x\to\infty} \frac{e^x}{x^2} = +\infty .$$

Ex: $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$

$$\begin{cases} \lim_{x\to\infty} \ln x = +\infty \\ \lim_{x\to\infty} \sqrt[3]{x} = +\infty \end{cases}$$ and $\frac{d}{dx}\sqrt[3]{x} = \frac{1}{3}x^{-2/3} \neq 0$ for x large

$$\text{So } \lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} \underset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} 3 \cdot \frac{x^{2/3}}{x} =$$

$$= \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0 .$$

Ex: Find $\lim_{x \to \pi^-} \frac{\sin x}{1-\cos x}$

Caution: Let's apply L'Hospital without checking hypothesis.

$$\lim_{x \to \pi^-} \frac{\sin x}{1-\cos x} = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = +\infty$$

But!

$$\lim_{x \to \pi^-} \frac{\sin x}{1-\cos x} = \frac{\lim_{x \to \pi^-} \sin x}{\lim_{x \to \pi^-} 1-\cos x} = \frac{0}{2} = 0$$

quotient rule for limits

L'Hospital's rule cannot be applied here because

$\lim_{x \to \pi} \sin(x) = 0$ but $\lim_{x \to \pi^-} 1-\cos x = 2$

Suppose you want to compute

$\lim_{x\to a} f\cdot g$ with $f(x)\to\pm\infty$, $g(x)\to 0$

$\boxed{\text{Indeterminate of type } 0\cdot\infty}$

Can instead write

$\lim_{x\to a} \frac{f}{1/g}$ and apply L'Hospital

Ex: Compute $\lim_{x\to 0^+} \sin(x)\cdot\ln(x)$

Write

$$\lim_{x\to 0^+} \frac{\ln x}{(\sin x)^{-1}}$$

Then $\begin{cases} \lim_{x\to 0^+} (\sin x)^{-1} = \infty \\ \lim_{x\to 0^+} \ln x = \infty \end{cases}$ and $\frac{d}{dx}(\sin x)^{-1} = -(\sin x)^{-2}\cos x$ $\neq 0$ for $x \approx 0$

So
$$\lim_{x\to 0^+} \frac{\ln x}{(\sin x)^{-1}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-(\sin x)^{-2}\cos x} =$$

$$= \lim_{x\to 0^+} -\frac{(\sin x)^2}{x\cos x} =$$

$$= \left[\lim_{x\to 0^+} -\frac{\sin x}{x}\right]\cdot\left[\lim_{x\to 0} \frac{\sin x}{\cos x}\right]$$

$$= (-1)\cdot 0 = 0.$$

Suppose we want to compute

$$\lim_{x\to a} [f(x)-g(x)]$$

and $\lim_{x\to a} f(x)=\infty$

and $\lim_{x\to a} g(x) = -\infty$.

$\boxed{\text{Indeterminate of type } \infty-\infty}$

<u>Strategy</u>: Rewrite $f(x)-g(x)$ as

a

<u>Ex</u>: $\lim_{x\to \frac{\pi}{2}^-} \sec x - \tan x$

$\lim_{x\to\frac{\pi}{2}^-} \sec x = -\infty$ and $\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty$

<u>Write</u>:

$$\lim_{x\to\frac{\pi}{2}^-} \sec x - \tan x = \lim_{x\to\frac{\pi}{2}^-}\left[\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} = 0$$

L'Hospital

(Check the hypothesis!)

- Several Indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1) $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ $\boxed{\text{type } 0^0}$

2) $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ $\boxed{\text{type } \infty^0}$

3) $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ $\boxed{\text{type } 1^\infty}$

- In all cases, the stategy is to take logs and then use L'Hospital.

Ex: Calculate $\lim_{x \to 0^+} (1 + \sin(4x))^{\cot x}$

Note $1 + \sin(4x) \to 1$ as $x \to 0^+$

$\frac{\cos x}{\sin x} = \cot x \to -\infty$ as $x \to 0^+$

Set $y = (1 + \sin(4x))^{\cot x}$

Then $\ln y = (\cot x) \ln(1 + \sin(4x))$

So

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} (\cot x) \ln(1+\sin 4x)$$

indeterminate of type $\infty \cdot 0$

$$= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{(\cot(x))^{-1}} = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x}$$

L'Hospital ↓

$$= \lim_{x\to 0^+} \frac{\left(\frac{1}{1+\sin 4x}\right)(4\cos 4x)}{\sec^2(x)} =$$

$$= \lim_{x\to 0^+} \frac{4\cos 4x}{(1+\sin 4x)\cdot \frac{1}{\cos^2 x}} = 4$$

Finally

$$\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = \lim_{x\to 0^+} y =$$

$$= \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y} =$$

$$= e^4.$$