Lecture 21: Midterm Review

Ex: Find the derivatives:

ⓐ $\frac{d}{dx} \sin(2+\sin(\sqrt{1+x^3})) =$

$= \cos(2+\sin\sqrt{1+x^3}) \cdot \frac{d}{dx}(2+\sin\sqrt{1+x^3})$

$= \cos(2+\sin\sqrt{1+x^3}) \cos\sqrt{1+x^3} \cdot \frac{d}{dx}\sqrt{1+x^3}$

$= \cos(2+\sin\sqrt{1+x^3}) \cos\sqrt{1+x^3} \cdot \frac{1}{2\sqrt{1+x^3}} \cdot 3x^2.$

ⓑ $f(x) = (\arctan x)^{\ln x}$

$y = (\arctan x)^{\ln x}$

$\ln y = \ln((\arctan x)^{\ln x}) = (\ln x)[\ln(\arctan x)]$

$\frac{y'}{y} = \frac{d}{dx}\ln y = \left[\frac{d}{dx}\ln x\right][\ln(\arctan x)] + \ln x \left[\frac{d}{dx}\ln(\arctan x)\right]$

$= \frac{\ln(\arctan x)}{x} + \ln x \cdot \frac{1}{\arctan x}\left[\frac{d}{dx}\arctan x\right]$

$= \frac{\ln(\arctan x)}{x} + \frac{\ln x}{(\arctan x)} \cdot \frac{1}{1+x^2}$

So $y' = (\arctan x)^{\ln x}\left[\frac{\ln(\arctan x)}{x} + \frac{\ln x}{(1+x^2)\arctan x}\right]$

Ex: Consider the parametric curve

$$x(t) = t^3 - 7t + 5$$
$$y(t) = 2t^3 - 3t^2 + 3t$$

(a) Find equation of tangent line to the curve when $t = -1$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{6t^2 - 6t + 3}{3t^2 - 7}$$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{6+6+3}{3-7} = \frac{15}{-4}$$

$$y(-1) = -2 - 3 - 3 = -8$$
$$x(-1) = -1 + 7 + 5 = 11$$

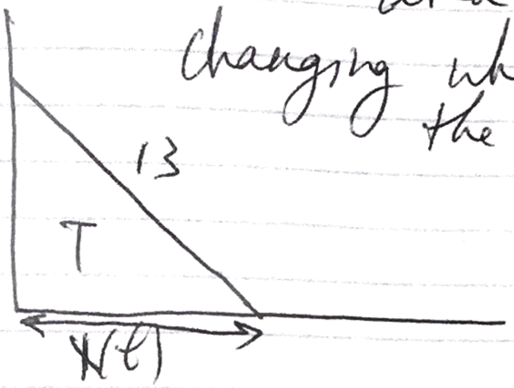
Tangent line:

$$y + 8 = -\frac{15}{4}(x - 11)$$

(b) Find all times t when the tangent line has slope 3.

Set
$$3 = \frac{dy}{dx} = \frac{6t^2 - 6t + 3}{3t^2 - 7} \iff 9t^2 - 21 = 6t^2 - 6t + 3$$
$$\iff 3t^2 + 6t - 24 = 0$$
$$\iff t^2 + 2t - 8 = 0$$
$$(t-2)(t+4) = 0$$
$$t = 2, \quad t = -4$$

Ex: A ladder 13 ft long rests against a vertical wall. The bottom of the ladder slides away from the wall at a rate of 0.5 ft/s. At what rate is the area of the triangle T changing when the bottom of the ladder is 12 ft from the wall



So $A(t) = \frac{1}{2} x(t)\left(\sqrt{13^2 - x^2(t)}\right)$

area at time t

$$A'(t) = \frac{1}{2}\frac{d}{dt}\left[x\sqrt{13^2-x^2}\right] =$$

$$= \frac{1}{2}\left[\left[\frac{d}{dt}x\right]\sqrt{13^2-x^2} + x\frac{d}{dt}\sqrt{13^2-x^2}\right)$$

$$= \frac{1}{2}\left[x'\sqrt{13^2-x^2} + x\frac{1}{2\sqrt{169-x^2}}\cdot(-2xx')\right]$$

Plug in $x'(t) = 0.5$, $x(t) = 12$, Note. $\sqrt{13^2-12^2} = 5$

$$A'(t) = \frac{1}{2}\left[\frac{1}{2}\cdot 5 + \frac{12}{2\cdot 5}\left(-2\cdot 12\cdot \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{5}{2} - \frac{72}{5}\right]$$

Ex: Use a linear approximation to estimate $\sqrt[5]{33}$.

Define

$f(x) = \sqrt[5]{x}$

Let's find a linear approximation of f at $a = 32$ because $f(a)$ and $f'(a)$ are easy to compute

$f(32) = \sqrt[5]{32} = 2$

$f'(x) = \frac{d}{dx} x^{1/5} = \frac{1}{5} x^{-4/5} = \frac{1}{5x^{4/5}}$

$f'(a) = \frac{1}{5 \cdot 2^4} = \frac{1}{80}$

So linear approximation of f at a is

$L(x) = f(a) + f'(a)(x-a)$

$= 2 + \frac{1}{80}(x-32)$

So

$\sqrt[5]{33} = f(33) \approx L(33) = 2 + \frac{1}{80}(33-32)$

$= 2 + \frac{1}{80}$.

Ex Find all points (a,b) on the curve $2x^3+y^2-8y=0$ where the tangent line is horizontal

$$\frac{d}{dx}\left[2x^3+y^2-8y\right]=0$$

$$6x^2+2yy'-8y'=0$$

Tangent vertical means $y'=0$.

$$\Leftrightarrow\ 6x=0 \Leftrightarrow x=0$$

Ex Consider the plane curve

$$y^2+7x^2y-3x^2=-[illegible]$$

ⓐ Compute the tangent line to the curve at $(2,-3)$

$$2yy'+7\frac{d}{dx}(x^2y)-6x=0$$

$$2yy'+7\left[2xy+x^2y'\right]-6x=0$$

Plug in $(2,-3)=(x,y)$

$$-6y'+7(-12+4y')-12=0$$

$$+22y'-84-12=0$$

$$y'=\frac{96}{+22}$$

Tangent line is $y+3=\frac{96}{22}(x-2)$

(b) Use the tangent line to approximate the y-value at a point near $(2,-3)$ where the x-coordinate is 2.1.

Tangent line at $(2,-3)$

$$y+3=\frac{96}{22}(x-2)$$

So the approximation is

$$y \approx -3+\frac{96}{22}(2.1-2) = -3+\frac{96}{220}$$

~~Consider the particle moving according to~~

$$\cancel{x(t)=4t^2+3t,\quad y(t)=-t^2+2}$$