Lecture 21: How derivatives affect the shape of a curve

Ex: Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing

Strategy: Find intervals on which $f' > 0$, $f' < 0$

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2) = 12x(x-2)(x+1)$$

So $f'(x) = 0 \iff x = 0, 2, -1$

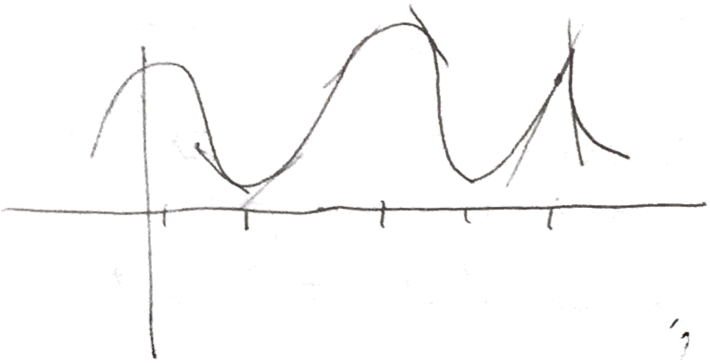
Interval	f'	f
$(-\infty, -1)$	$-$	decreasing
$(-1, 0)$	$+$	increasing
$(0, 2)$	$-$	decreasing
$(2, +\infty)$	$+$	increasing

Recall local maximizer/minimizer $x \Rightarrow f'(x) = 0$ $\not\Leftarrow$

Thm: Suppose that c is a critical number of a continuous function. Then

① If f' changes from positive to negative to c, then f has a local ~~minimum~~ maximum at c

② If f' changes from negative to positive to c, then f has a local ~~maximum~~ minimum at c.

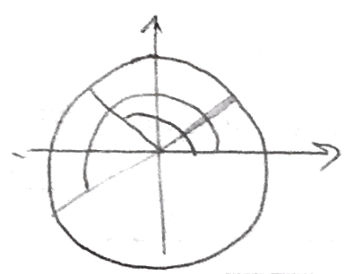


In previous example, $x=-1 \Leftarrow$ local minimum
$x=0 \Leftarrow$ local maximum
$x=2 \Leftarrow$ local minimum

<u>Ex:</u> Find the local minimum and maximum values of

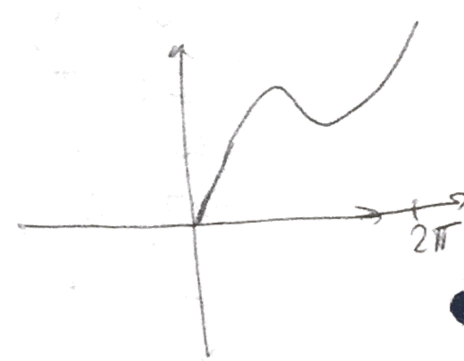
$$f(x) = x + 2\sin x, \quad 0 \le x \le 2\pi$$

$$f'(x) = 1 + 2\cos x = 0 \Leftrightarrow \cos x = -\frac{1}{2}$$



$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Interval	g'	g
$(0, \frac{2\pi}{3})$	+	increasing
$(\frac{2\pi}{3}, \frac{4\pi}{3})$	−	decreasing
$(\frac{4\pi}{3}, 2\pi)$	+	increasing



$\Rightarrow$ $\frac{2\pi}{3} \Leftarrow$ local maximizer
$\frac{4\pi}{3} \Leftarrow$ local minimizer

Ex: find local minima/maxima of $f(x) = \dfrac{x^2}{x-1}$

Domain $= (-\infty, 1) \cup (1, \infty)$

$$f'(x) = \frac{(x-1)\frac{d}{dx}x^2 - \left[\frac{d}{dx}(x-1)\right]x^2}{(x-1)^2} =$$

$$= \frac{2x(x-1) - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

Critical points are

and $x(x-2) = 0 \iff x = 0,\ x = 2$

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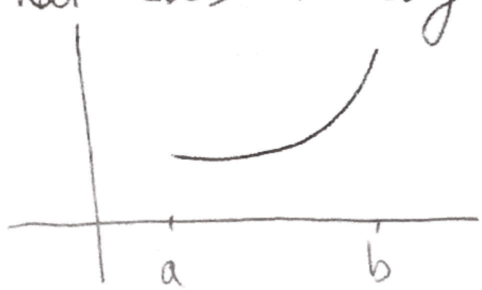
Note: The function is not defined at $x =$ [illegible], so it is neither a local max or local min

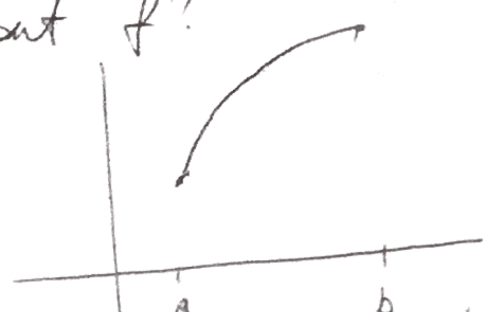
Interval	f'	f
$(-\infty, 0)$	$+$	increasing
$(0, 2)$	$-$	decreasing
$(2, \infty)$	$+$	increasing

So $x = 0 \leftarrow$ local maximum
$x = 2 \leftarrow$ local minimum.

What does f'' say about f?

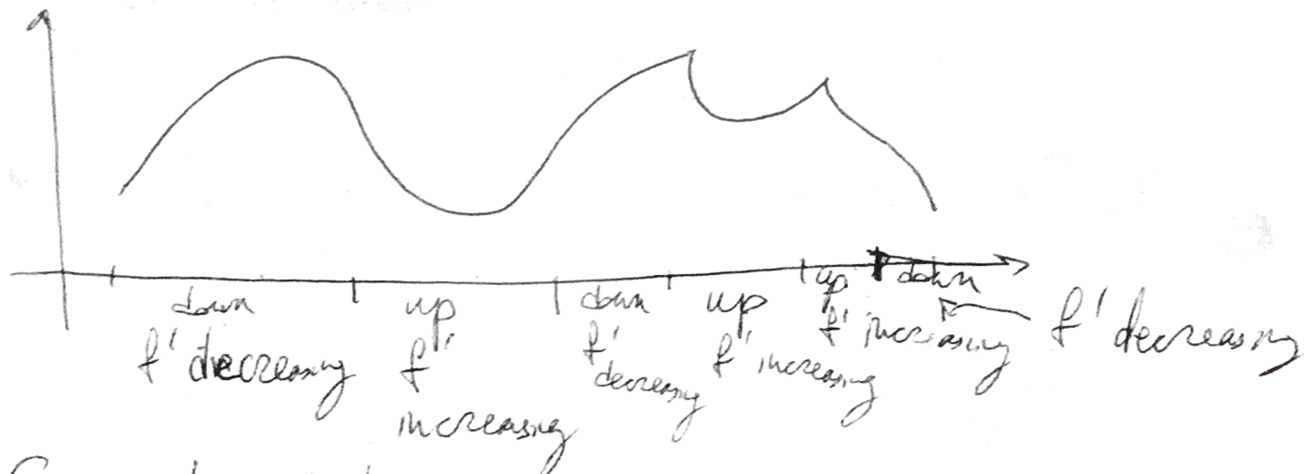


concave up



concave down

Defn: If the graph of f lies above all its tangent lines on an interval I, then it is called concave up on I. If the graph of f lies below all of its tangent lines on I, it is called concave down on I



Concavity Test

(a) If $f''(x) > 0$ for all x in I, then f is concave up on I

(b) If $f''(x) < 0$ for all x in I, then f is concave down on I

Definition: A point c is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave down to up at c.

Thm Suppose f'' is continuous at c. Then c is an inflection point $\Rightarrow$ ~~and only~~ $f''(c) = 0$

Ex: $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

Inflection points: $x = 0, 2$.

interval	f''	Concavity
$(-\infty, 0)$	+	up
$(0, 2)$	−	down
$(2, \infty)$	+	up