Lecture 20: Finding minima and maxima

- So if f has a local maximum or minimum at c, then c is a critical number of f

The closed interval method

To find an absolute maximum or minimum values of a continuous function f on a closed interval $[a,b]$

1) Find the critical values of f on (a,b)
2) Find $f(a)$, $f(b)$
3) The largest of the values from Steps 1 and 2 is the absolut maximum value; the smallest of these values is the absolute minimum value.

- Ex: Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1, \quad -\tfrac{1}{2} \le x \le 4$$

Notice f is continuous

$$f'(x) = 3x^2 - 6x = 0 \iff 3x(x-2) = 0$$

$$x = 0 \quad \text{or} \quad x = 2$$

So candidates are $x = \underbrace{0, 2}_{\text{critical points}}, \underbrace{-\tfrac{1}{2}, 4}_{\text{endpoints}}$

- $$\left.\begin{aligned} f(0) &= 1 \\ f(2) &= 8 - 3\cdot 4 + 1 = -3 \\ f(-\tfrac{1}{2}) &= \tfrac{1}{8} \\ f(4) &= 17 \end{aligned}\right\} \Rightarrow \begin{aligned} &\text{absolute minimum} = -3 \\ &\text{absolute maximum} = 17 \end{aligned}$$

Ex: Find the absolute minimum and maximum of

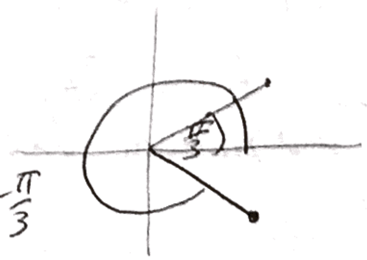
$$f(x) = x - 2\sin x \quad \text{on} \quad 0 \le x \le 2\pi$$

f is differentiable (so continuous)

$$f'(x) = 1 - 2\cos(x) = 0 \iff \cos(x) = \frac{1}{2}$$

$$\iff x = \frac{\pi}{3}, \left(-\frac{\pi}{3} + 2\pi\right) = \frac{5}{3}\pi$$

$2\pi - \frac{\pi}{3}$



So candidates are

$$x = \underbrace{\frac{\pi}{3}, \frac{5}{3}\pi}_{\text{critical points}}, \underbrace{0, 2\pi}_{\text{endpoints}}$$

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2\sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \frac{2\sqrt{3}}{2} = \frac{\pi}{3} - \sqrt{3} \approx -0.68485$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3} \approx 6.968$$

$$f(0) = 0 - 2\sin(0) = 0$$

$$f(2\pi) = 2\pi - 2\sin(2\pi) = 2\pi \approx 6.28$$

So absolute minimum $= \frac{\pi}{3} - \sqrt{3}$

absolute maximum $= \frac{5\pi}{3} + \sqrt{3}$

Ex: Find the absolute maximum and minimum of

$$f(x) = \frac{x}{x^2 - x + 1} \quad \text{on } [0,3]$$

Is f continuous?

does not exist

$$x^2 - x + 1 = 0 \iff x = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2}$$

no roots.

So can use the closed interval method

$$f'(x) = \frac{(x^2-x+1)\frac{d}{dx}x - \left[\frac{d}{dx}(x^2-x+1)\right]x}{(x^2-x+1)^2} =$$

$$= \frac{x^2-x+1-(2x-1)x}{(x^2-x+1)^2} = \frac{-x^2+1}{(x^2-x+1)^2}$$

$$f'(x) = 0 \iff -x^2+1 = 0 \iff x = \pm 1$$

So candidates $x = \underbrace{1, \cancel{-1}}_{\text{critical points}}\ \underbrace{0, 3}_{\text{endpoints}}$

$$f(1) = \frac{1}{1-1+1} = 1$$

$$\cancel{f(-1) = \frac{-1}{(-1)^2-(-1)+1} = -\frac{1}{3}}$$

$$f(0) = 0$$

$$f(3) = \frac{3}{9-3+1} = \frac{3}{7}$$

$\Rightarrow$ ~~absolute minimum $= -1/3$~~

absolute maximum $= 1$

absolute minimum $= 0$.

$\underline{EX}$: Find the absolute maximum and minimum of $f(x) = x\sqrt{x-x^2}$ on the real line.

What is the domain of f.

$$x - x^2 \geq 0 \iff x(1-x) \geq 0$$

$$\iff x \geq 0 \text{ and } x \leq 1$$

or

$$x \leq 0 \text{ and } 1 - x \leq 0$$

$$\iff x \text{ is in } [0,1]$$

domain of f.

f is continuous on $[0,1]$

$$f'(x) = \left[\frac{d}{dx} x\right]\sqrt{x-x^2} + x\left[\frac{d}{dx}\sqrt{x-x^2}\right]$$

$$= \sqrt{x-x^2} + x \cdot \frac{1}{2}(x-x^2)^{-1/2} \cdot (1-2x)$$

$$= \sqrt{x-x^2} + \frac{x(1-2x)}{2\sqrt{x-x^2}} = \frac{2(x-x^2)}{2\sqrt{x-x^2}} + \frac{x(1-2x)}{2\sqrt{x-x^2}} =$$

$$= \frac{3x - 4x^2}{2\sqrt{x-x^2}}$$

$f'(x) = 0 \iff x - 4x^2 = 0 \iff x(3-4x) = 0$ and x in $(0,1)$

and x in $(0,1)$ $\quad$ x in $(0,1)$ $\quad \iff x = \frac{3}{4}$

So candidates $x = 0, \frac{3}{4}, 1$

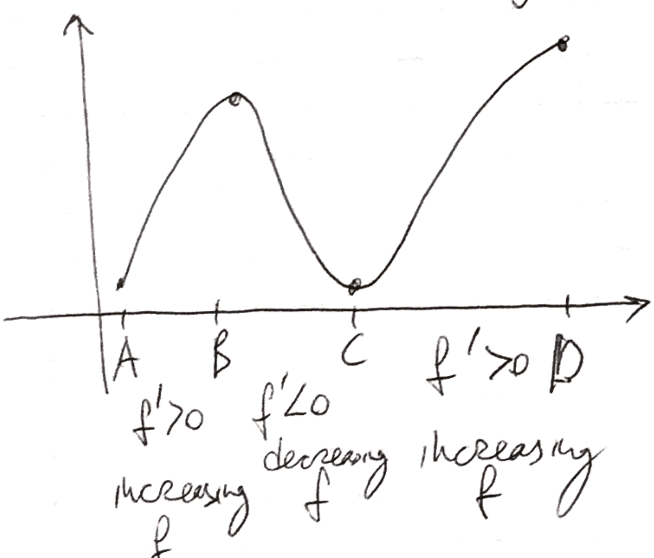
$\Rightarrow f(0) = 0,\ f(3/4) = \frac{3\sqrt{3}}{16},\ f(1) = 0 \Rightarrow$ absolute min $= 0$, absolute max $= \frac{3\sqrt{3}}{16}$.

How derivatives affect the shape of a curve

(I) What does f' say about f?

Defn: A function f is increasing on an interval I if $f(x) < f(y)$ whenever $x < y$ in I.

A function f is decreasing on I if $f(x) > f(y)$ whenever $x < y$ in I.



Increasing/Decreasing Test

① If $f'(x) > 0$ on an interval, then f is increasing on that interval

② If $f'(x) < 0$ on an interval, then f is decreasing on that interval