Lecture 2: Parametric Curves and Rates of Change

i) **Parametric Curves**

**Definition:** Given functions \( f \) and \( g \), the curve traced out by the points \( t \mapsto (f(t), g(t)) \) is called a parametric curve.

**Example:**

\[
x = f(t) = 5t + 1
\]
\[
y = g(t) = 5t - 1
\]

Sketching: \( x^2 - y^2 = 2 \) with \( x \geq \sqrt{2}, \ y \geq 0 \)

**Example:**

\[
x = \sin(2t)
\]
\[
y = \cos(2t)
\]

Notice \( x^2 + y^2 = 1 \)

The particle moves around the circle clockwise making two full rotations.
Motivation of the limit.

Consider function \( f : \mathbb{R} \rightarrow \mathbb{R} \). and two points \( a, b \).

If \( f \) describes the position of an object on the real line, the quantity \( \frac{f(b) - f(a)}{b-a} \) is called the average velocity from \( t=a \) to \( t=b \).

What is the instantaneous velocity at \( a \)?

Instantaneous velocity at \( a \) is \( \lim_{b \to a} \frac{f(b) - f(a)}{b-a} \).

= slope of the "tangent line" to \( f \) at \( a \).

Eqn at \( b \): \( y = m(x-a) + f(a) \)

\( m \) is instantaneous velocity.
Limit of a function

Write \( \lim_{x \to a} f(x) = L \) (read "limit of \( f \) as \( x \) approaches \( a \) equals \( L \)"")

if we can make the values \( f(x) \) arbitrarily close to \( L \) by taking \( x \) sufficiently close to \( a \) (on either side of \( a \)) but not equal to \( a \).

Also write \( f(x) \to L \) as \( x \to a \).

\[
\lim_{x \to a} f(x) = L \quad \text{Note: We don't care about \( f(a) \).}
\]

\[
\lim_{x \to a} f(x) \text{ does not exist}
\]

In all three cases \( \lim_{x \to a} f(x) = L \).
How to find the limit?

**Brute-force method:**

Let $x \to a$ from both sides.

Tabulate $f(x)$ as $x \to a$ and see if the limit leads to a fixed value.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\frac{\sin x}{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pm 1.0$</td>
<td>0.84147</td>
</tr>
<tr>
<td>$\pm 0.5$</td>
<td>0.95885</td>
</tr>
<tr>
<td>$\pm 0.4$</td>
<td>0.973545</td>
</tr>
<tr>
<td>$\pm 0.3$</td>
<td>0.98506</td>
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<tr>
<td>$\pm 0.2$</td>
<td>0.99334</td>
</tr>
<tr>
<td>$\pm 0.1$</td>
<td>0.99833</td>
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<tr>
<td>$\pm 0.05$</td>
<td>0.99958</td>
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<tr>
<td>$\pm 0.01$</td>
<td>0.99993</td>
</tr>
<tr>
<td>$\pm 0.005$</td>
<td>0.999993</td>
</tr>
<tr>
<td>$\pm 0.001$</td>
<td>0.999999</td>
</tr>
</tbody>
</table>

We guess $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Ex: What is $\lim_{x \to 0} \frac{\sin \pi x}{x}$?

If $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$, then $\sin \pi x = 0$.

But for infinitely many $x$ near $0$, we have $\sin \pi x = 0$.

$\Rightarrow \lim_{x \to 0} \frac{\sin \pi x}{x}$ does not exist.
One-sided limits

\( \lim_{x \to a^-} f(x) = L \) means \( f(x) \) approaches \( L \) as \( x \to a^- \).

Similarly,

\( \lim_{x \to a^+} f(x) = L \) means \( f(x) \) approaches \( L \) as \( x \to a^+ \).

\[ \begin{align*}
\lim_{x \to 2^-} f(x) &= -1, & \lim_{x \to 2^+} f(x) &= 1 \\
\therefore \lim_{x \to 2} f(x) \text{ does not exist. Because} & \quad \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \\
\lim_{x \to 2^-} f(x) &= 1 & \lim_{x \to 2} f(x) \text{ does not exist} \\
\lim_{x \to 2^+} f(x) &= 2 &
\end{align*} \]

2 = \lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = \lim_{x \to 4} f(x) = 2.
Infinite limits

Definition: \( \lim_{x \to a} f(x) = +\infty \)

\( f(x) \) can be made arbitrarily large as \( x \to a \).

The term \( \lim_{x \to a} f(x) = -\infty \) is defined similarly.

Ex: \( \lim_{x \to a^+} f(x) = +\infty \) or \( \lim_{x \to a^-} f(x) = -\infty \)

Ex: \( \lim_{x \to 0^+} \ln x = -\infty \)

So \( x = 0 \) is a vertical asymptote.

Ex: Find the vertical asymptotes of \( f(x) = \tan x \)

So vertical asymptotes are at \( x = \left(\frac{k\pi}{2}\right) \) where \( k \) is an integer.