Lecture 2: Parametric Curves and Rates of Change

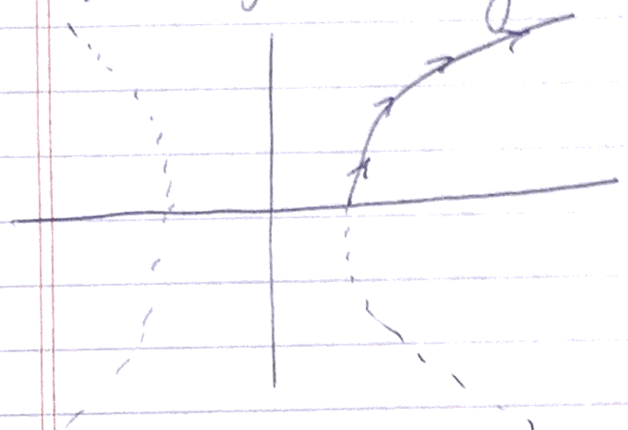
① Parametric Curves

<u>Defn</u>: Given functions f and g, the curve traced out by the points $t \mapsto (f(t), g(t))$ is called a <u>parametric curve</u>

<u>Ex</u>: $x = f(t) = \sqrt{t+1}$ for $t \geq 1$

$y = g(t) = \sqrt{t-1}$

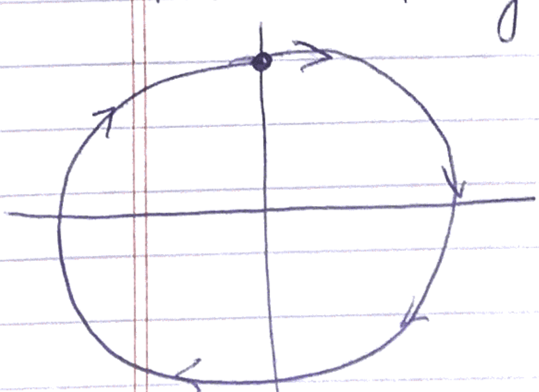
<u>Sketching</u>: $x^2 - y^2 = 2$ with $x \geq \sqrt{2}$, $y \geq 0$



<u>Ex</u>: $x = \sin(2t)$ for $0 \leq t \leq 2\pi$

$y = \cos(2t)$

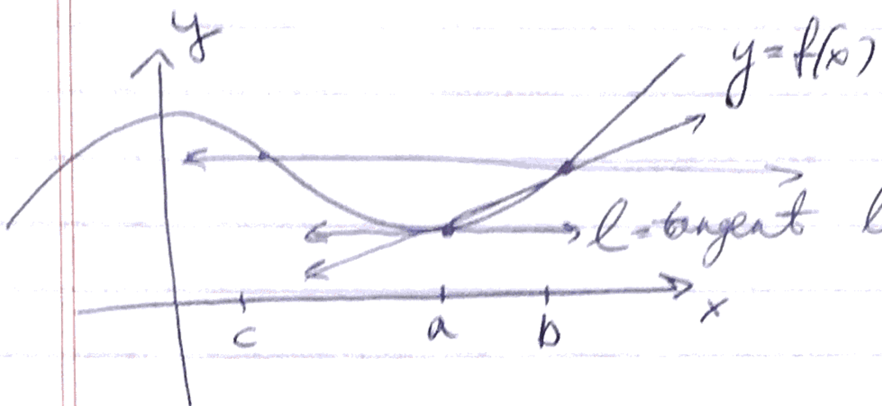
Notice $x^2 + y^2 = 1$



The particle moves around the circle clockwise making two full rotations.

Motivation of the limit

Consider function $f: \mathbb{R} \to \mathbb{R}$. and two points a, b.



If $f(t)$ describes the position of an object on the real line, the quantity

$$\frac{f(b)-f(a)}{b-a}$$

is called the <u>average velocity</u> from $t=a$ to $t=b$.

What is the <u>instantaneous velocity</u> at a?

instantaneous velocity at a := $\lim_{b \to a} \frac{f(b)-f(a)}{b-a}$

= slope of the "tangent line" to f at a.

Eqn of l: $y = m(x-a) + f(a)$

instantaneous velocity

Limit of a function

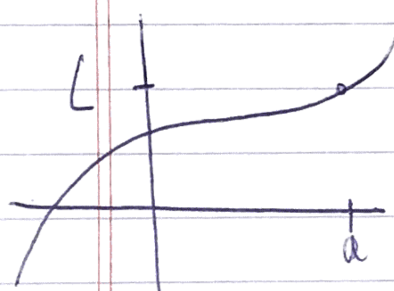
Write $\lim_{x \to a} f(x) = L$ (read "limit of f as x approaches a equals L")

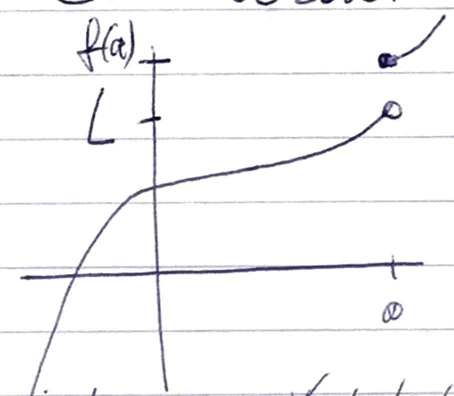
if we can make the values $f(x)$ arbitrarily close to L by taking x sufficiently close to a (on either side of a) but not equal to a.

Also write $f(x) \to L$ as $x \to a$.

Note: We don't care about $f(a)$.

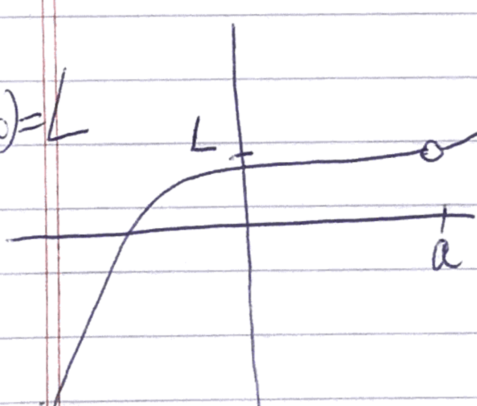
$\lim_{x \to a} f(x) = L$





$\lim_{x \to a} f(x)$ does not exist

$\lim_{x \to a} f(x) = L$



~~In all three cases $\lim_{x \to a} f(x) \neq L$.~~

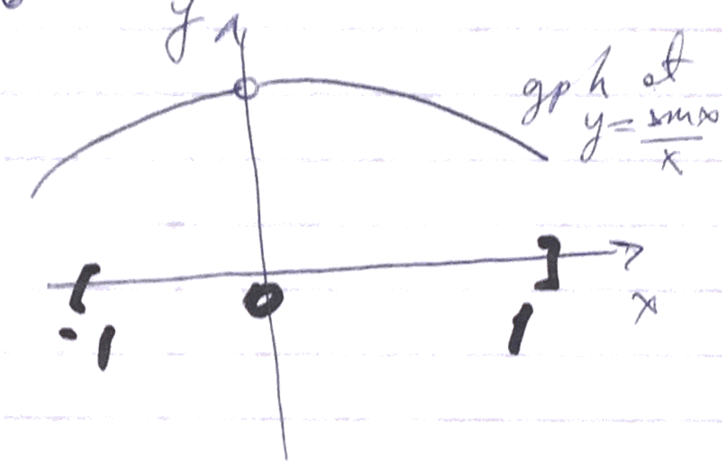
How to find the limit?

Brute-force method:

Let $x \to a$ from both sides.
Tabulate $f(x)$ as $x \to a$ and see if the tend to a fixed value.

Ex:

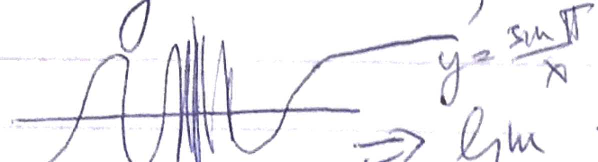
x	$\frac{\sin x}{x}$
± 1.0	0.84147
± 0.5	0.95885
± 0.4	0.973545
± 0.3	0.98506
± 0.2	0.99334
± 0.1	0.99833
± 0.05	0.99958
± 0.01	0.999983
± 0.005	0.999995
± 0.001	0.999999



We guess $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Ex: What is $\lim_{x \to 0} \sin \frac{\pi}{x}$?

If $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ then $\sin \frac{\pi}{x} = 0$
but for infinitely many x near 0, we have
$\sin \frac{\pi}{x} = 1$



$\Rightarrow \lim_{x \to 0} \frac{\sin x}{x}$ does not exist.

One-sided limits

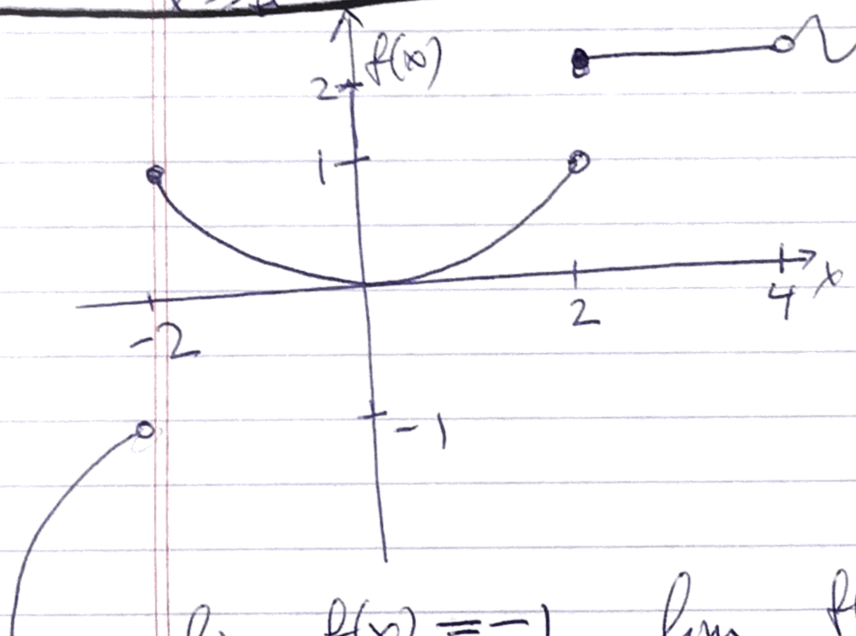
$x \to a^-$ if x approaches a from the left

$x \to a^+$ if x approaches a from the right.

$\lim\limits_{x \to a^-} f(x) = L$ means $f(x)$ approaches L as $x \to a^-$.

Similarly

$\lim\limits_{x \to a^+} f(x) = L$ " " " $x \to a^+$.



Observation: $\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = L$

$\lim\limits_{x \to -2^-} f(x) = -1$, $\lim\limits_{x \to -2^+} f(x) = 1$

$\therefore \lim\limits_{x \to -2} f(x)$ does not exist because $\lim\limits_{x \to -2^-} f(x) \neq \lim\limits_{x \to -2^+} f(x)$.

$$\left.\begin{array}{l} \lim\limits_{x \to 2^-} f(x) = 1 \\ \lim\limits_{x \to 2^+} f(x) = 2 \end{array}\right\} \Rightarrow \lim\limits_{x \to 2} f(x) \text{ does not exist}$$

$2 = \lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^+} f(x) \Rightarrow \lim\limits_{x \to 4} f(x) = 2.$

Infinite limits

Def: $\lim_{x \to a} f(x) = +\infty$

if $f(x)$ can be made arbitrarily large as $x \to a$

The term $\lim_{x \to a} f(x) = -\infty$ is defined similarly.

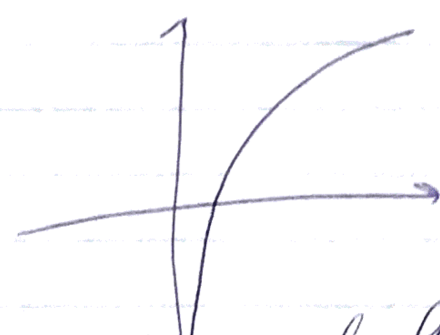
~~Ex~~:

Defn: The line $x = a$ is called a vertical asymptote if either

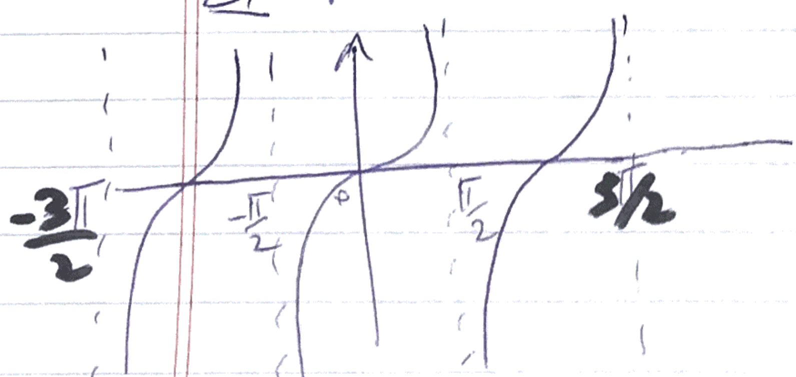
$\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$

Ex: $\lim_{x \to 0^+} \ln x = -\infty$

So $x = 0$ is a vertical asymptote



Ex: Find the vertical asymptotes of $f(x) = \tan x$.



So vertical assymptotes are at $x = \frac{(2k+1)\pi}{2}$ where k is an integer.