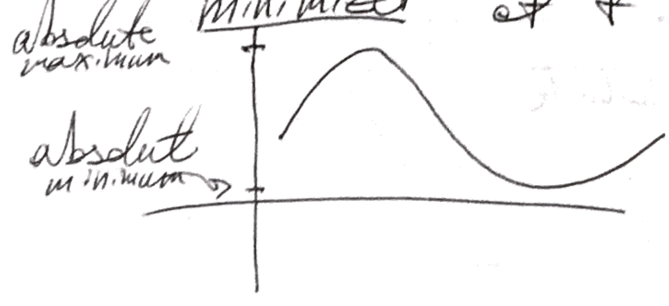
Lecture 19: Maximum and Minimum Values

__Goal:__ "Understand" minimum/maximum of a function f.

__Defn:__ Let c be a number in the domain D of a function f. Then $f(c)$ is the

- __absolute maximum (value)__ of f on D if $f(c) \geq f(x)$ for all x in D. If this is the case, we say c is a __global maximizer__ of f on D
- __absolute minimum (value)__ of f on D if $f(c) \leq f(x)$ for all x in D. If this is the case we say c is a __global minimizer__ of f on D.

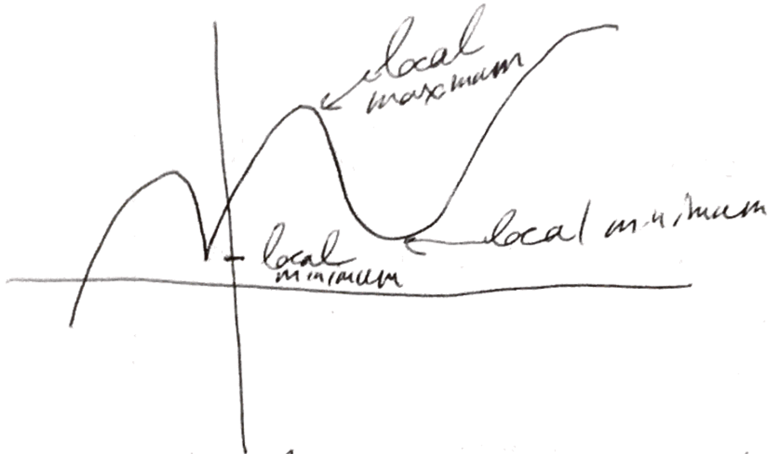


__Defn:__ The number $f(c)$ is a

- __local maximum (value)__ if $f(c) \geq f(x)$ for all x near c
- __local minimum (value)__ if $f(c) \leq f(x)$ for all x near c.

Note here "a property holds near c" means it holds on some open interval around c.

minima/maxima are also called extrema.



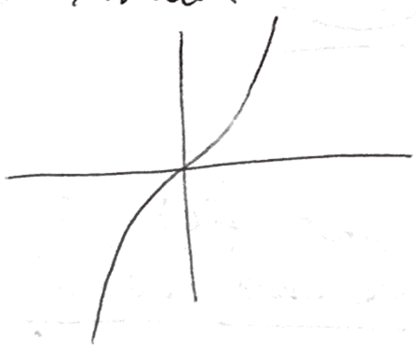
local minimum ⇐ absolute minimum
local maximum ⇐ absolute maximum

Ex: $f(x) = \cos(x)$.
absolute minimal value $= -1$
absolute maximum value $= 1$

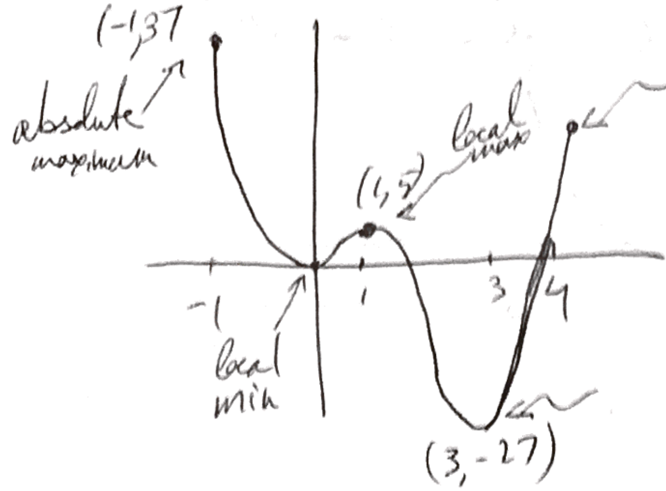
every local minimum is absolute
" maximum " "

Ex: $f(x) = x^2$
absolute minimum $= 0$ (at $x=0$)
no absolute maximum
local minimum is absolute
no local maximum

Ex: $f(x) = x^3$
no local extrema



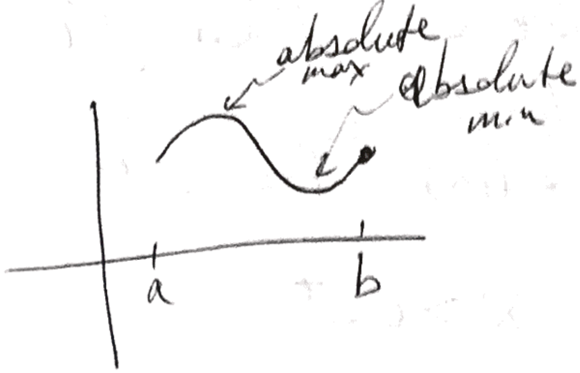
Ex: $f(x) = 3x^4 - 16x^3 + 18x^2 \quad -1 \le x \le 4$

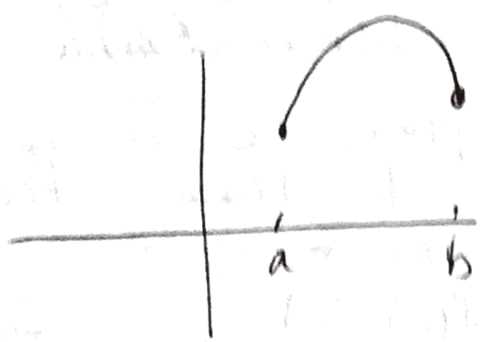


local max
(book claims it is not, but it is).

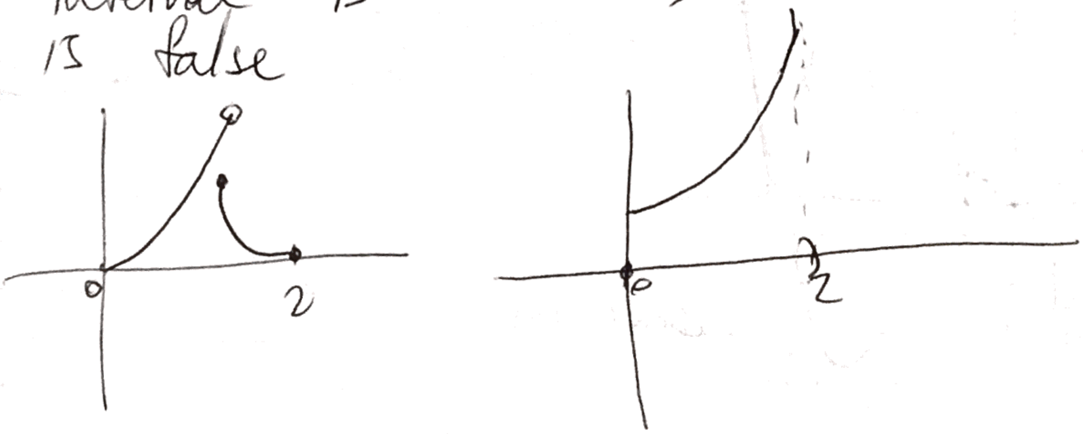
min

Thm: (The Extreme Value Theorem)
If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ on $[a,b]$





If f is not continuous on the interval is not closed, then theorem is false



Goal: Find derivative-based conditions that allow us to check if a point c is a local minimizer / maximizer

Thm: If c is a local minimizer or maximizer of a function f that is differentiable at c, then $f'(c) = 0$.

pf: Suppose c is a local minimizer of f. Then $f(c) \leq f(x)$ for all x near c.

$$\Rightarrow \frac{f(x) - f(c)}{x - c} \geq 0 \quad \text{for } x \to c^+$$

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0$$

On the other hand $\frac{f(x) - f(c)}{x - c} \leq 0$ for $x \to c^-$

$$\Rightarrow f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0$$

$\Rightarrow f'(c) = 0$. Similar argument for local maximum

Ex: $f(x) = x^2$

$f'(0) = 2 \cdot 0 = 0.$

So here indeed $f'(0) = 0$ at the minimizer 0.

Ex: $f(x) = x^3$

$f'(x) = 3x^2$

So $f'(0) = 0$, but $x = 0$ is not a local minimizer/maximizer of f!

Ex: $f(x) = |x|$ has an absolute minimum at 0 but is not differentiable at 0. So the theorem does not apply.

Def: A critical number of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist. The value $f(c)$ is called a critical value of f.

minima/maxima are also called extrema.

Ex: Find critical numbers of

$$f(x) = x^{3/5}(4-x)$$

product rule

$$f'(x) = \left[\frac{d}{dx} x^{3/5}\right](4-x) + x^{3/5}\left[\frac{d}{dx}(4-x)\right]$$

$$= \frac{3}{5}x^{-2/5}(4-x) - x^{3/5}$$

$$= \frac{3(4-x)}{5x^{2/5}} - x^{3/5} = \frac{12-3x}{5x^{2/5}} - \frac{5x}{5x^{2/5}}$$

$$= \frac{12-8x}{5x^{2/5}}$$

So critical numbers are numbers c where $f'(c)$ does not exist or $f'(c)=0$

$\Rightarrow$ $f'(c)$ does not exist $\Leftrightarrow 5c^{2/5}=0$

$\Leftrightarrow c=0.$

$f'(c)=0 \Leftrightarrow 12-8c=0 \Leftrightarrow c=\frac{12}{8}=\frac{3}{2}$