Lecture 18: Linear Approximation and Differentials

Consider a differentiable function f.
The tangent line to $y=f(x)$ at a is

$$y - f(a) = f'(a)(x-a)$$

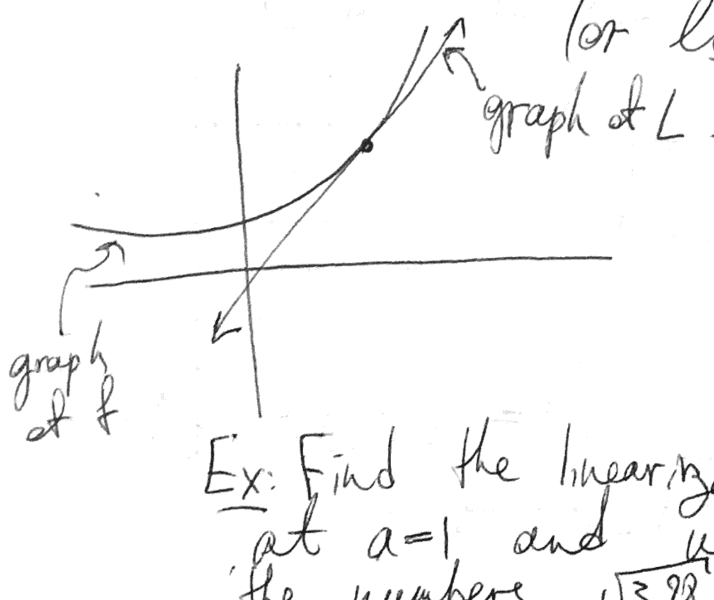
or equivalently

$$y = f(a) + f'(a)(x-a)$$

The Linear function whose graph is the tangent line

$$L(x) = f(a) + f'(a)(x-a)$$

is called the <u>linear approximation</u> of f at a.
(or <u>linearization</u>)



<u>Ex</u>: Find the linearization of $f(x) = \sqrt{x+3}$ at $a=1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

$f'(x) = \frac{1}{2}(x+3)^{-1/2} = \frac{1}{2\sqrt{x+3}}$

So $f'(1) = \frac{1}{4}$

So the linearization of f at $a=1$ is

$L(x) = \frac{1}{4}(x-1) + 2 = +\frac{7}{4} + \frac{x}{4}$

$\sqrt{4.05} \approx L(1.05) = \frac{7}{4} + \frac{1.05}{4} = 2.0125$
$\parallel$
$f(1.05)$

$\sqrt{3.98} \approx L(0.98) = \frac{7}{4} + \frac{0.98}{4} = 1.995.$
$\parallel$
$f(0.98)$

Ex: Find the linearization $L(x)$ of f at a.

(a) $f(x) = \sin x$, $a=0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

So $L(x) = f(0) + f'(0)(x-0)$
$= x$

(b) $f(x) = e^x \cos(x)$, $a=0$

$f'(x) = \left[\frac{d}{dx} e^x\right] \cos x + e^x \left[\frac{d}{dx} \cos x\right] = e^x \cos(x) - e^x \sin(x)$

$f'(0) = 1 \Rightarrow L(x) = f(0) + f'(0)(x-0)$
$= 1 + x$

Not part of curriculum: Taylor polynomials

Suppose we want to approximate f by a higher order polynomial:

$$f(x) \approx \underbrace{c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n}_{=P(x)}$$

Reasonable to ask that

$$f(a) = P(a) \Rightarrow c_0 = f(a)$$
$$f'(a) = p'(a)$$
$$f''(a) = p''(a)$$
$$\vdots$$
$$f^{(n)}(a) = p^{(n)}(a)$$

$$p'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1}$$
$$p'(a) = f'(a) \Rightarrow c_1 = f'(a)$$
$$p''(x) = 2c_2 + \cdots + n\cdot(n-1)(x-a)^{n-1}$$
$$p''(a) = f''(a) \Rightarrow 2c_2 = f''(a) \Rightarrow c_2 = \frac{1}{2}f''(a)$$

$$p^{(n)}(x) = n\cdot(n-1)\cdots 2\cdot 1\cdot c_n$$
$$p^{(n)}(a) = f^{(n)}(a) \Rightarrow n!\cdot c_n = f^{(n)}(a) \Rightarrow c_n = \frac{f^{(n)}(a)}{n!}$$

So best n-degree polynomial approximation of f at a is

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Degree n Taylor Polynomial

Ex: Find the quadratic approximation to

$f(x) = \sqrt{x+3}$ near $a = 1$

$$P(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$$

$$f(1) = \sqrt{1+3} = 2$$

$$f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$$

$$f''(1) = \frac{d}{dx}\frac{1}{2\sqrt{x+3}}\Big|_{x=1} = \frac{d}{dx}\frac{1}{2}(x+3)^{-1/2}\Big|_{x=1} =$$

$$= -\frac{1}{4}(x+3)^{-3/2}\Big|_{x=1}$$

$$= -\frac{1}{4(1+3)^{3/2}} = -\frac{1}{32}$$

So $P(x) = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2$.