Lecture 17: Related Rates

Related Rates Problems:

Goal: Compute the rate of change one quantity in terms of the rate of change of another related quantity which may be more easily measured

Ex: Air is pumped into a spherical balloon so that its volume increases at a constant rate of 100 cm³/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Soln: Let $V(t)$ be the volume of the ballon at time t.

$r(t)$ " " radius " " " "

" "

Key Equation:

$$V(t) = \frac{4}{3}\pi [r(t)]^3$$

Known: $V'(t)$ Unknown: $r'(t)$

$$\frac{d}{dt} V = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

$$V' = 4\pi r^2 r'$$

So $r' = \dfrac{V'}{4\pi r^2}$

Plug in $V' = 100$ and $r = 25$

$$\Rightarrow r' = \frac{100}{4\pi (25)^2} = \frac{1}{25\pi} \text{ cm/sec}$$

General Strategy:

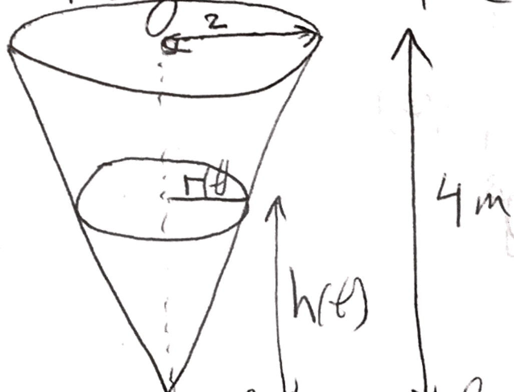
1) Record known and unknown information

2) Introduce mathematical notation for relevant quantities

3) Form an equation relating the quantities

4) Differentiate and solve for unknown

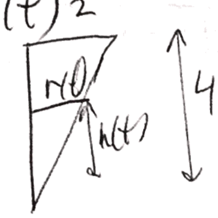
Ex: A water tank has the shape of an inverted circular cone with base radius 2m and height 4 m. If water is pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep



Strategy relate volume at time t (call it $V(t)$) to the height of the water level at time t (call it $h(t)$).

$$V(t) = \frac{1}{3}\pi [r(t)]^2 h(t)$$

Using similar triangles



we get $\frac{r}{h} = \frac{2}{4} \Rightarrow r = \frac{1}{2}h$

So $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$

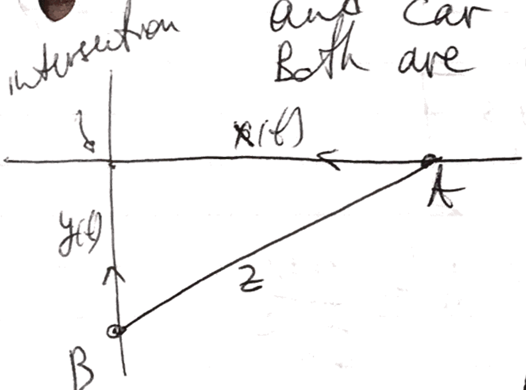
$\frac{d}{dt}V = \frac{d}{dt}\frac{1}{12}\pi h^3$

$V' = \frac{1}{4}\pi h^2 \cdot h'$

So $h' = \frac{4V'}{\pi h^2}$

Plug in $h = 3$, $V' = 2$ $\Rightarrow h' = \frac{4 \cdot 2}{\pi (3)^2} = \frac{8}{9\pi}$ m/min

Ex: Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection



Given: $x'(t) = -50$, $y'(t) = -60$

Task: Find $z'(t)$,

$z^2 = x^2 + y^2$

So $\frac{d}{dt}z^2 = \frac{d}{dt}(x^2+y^2)$

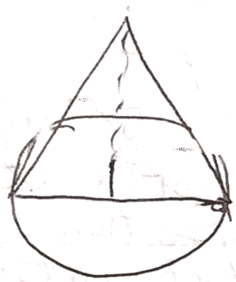
$\cancel{2}zz' = \cancel{2}xx' + \cancel{2}yy'$

Plug in: $x' = -50$, $y' = -60$, $x = 0.3$, $y = 0.4$, $z = \sqrt{0.3^2+0.4^2}$

$\frac{1}{2}z' = (-50)(0.3) + (0.4)(-60) \Rightarrow z' = -78$ mi/h

Ex: Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min in such a way that it forms a pile in a shape of a cone whose diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

$d(t)$: diameter at time t.
$h(t)$: height at time t.
$V(t)$: volume at time t.



$$V = \frac{1}{3}\pi\left(\frac{d}{2}\right)^2 h, \quad d(t) = h(t)$$

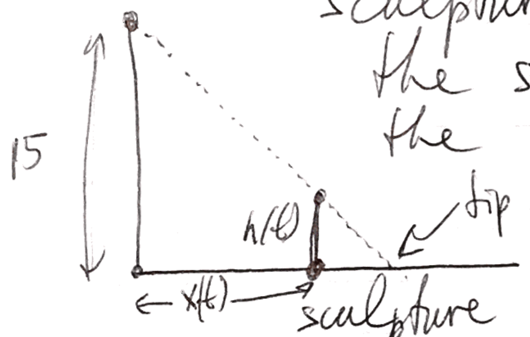
So $$\frac{d}{dt}V = \frac{d}{dt}\left[\frac{1}{12}\pi h^3\right]$$

$$V' = \frac{1}{4}\pi h^2 h' \Rightarrow h' = \frac{4V'}{\pi h^2}$$

Plug in $h=10$, $V'=30$

$$\Rightarrow \cancel{h' = \frac{4\cdot 30}{\pi(100)}} \quad \frac{6}{5\pi} = h'$$

Ex: A street light is mounted at the top of a 15 ft tall pole. An ice sculpture is being moved away from the pole at a speed of 3 ft/min along a straight path. The sculpture's height is decreasing at a rate of 0.2 ft/min. How fast is the tip of the sculpture's shadow moving, when the sculpture is 10 ft from the pole and has height 5 ft?



Let d(t) be the distance of the sculpture's shadow's tip to the pole. Then

$$\frac{d(t)}{15} = \frac{d(t)-x(t)}{h(t)}$$

$$\frac{d}{dt}\frac{d(t)}{15} = \frac{d}{dt}\left[\frac{d(t)-x(t)}{h(t)}\right]$$

$$\frac{d'}{15} = \frac{h(d'-x') - h'\cdot(d-x)}{h^2}$$

Plug in $x'=3$, $h'=-0.2$, $x=10$, $h=5$

$\frac{d}{15} = \frac{d-10}{5} \Rightarrow d=15$

$$\Rightarrow \frac{d'}{15} = \left[\frac{5(d'-3)+0.2(5)}{25}\right] \Rightarrow d' = \frac{42}{10}$$