Lecture 16: Implicit Differentiation Cont'd

Derivatives of Trig Functions

Recall

$y = \sin^{-1}(x)$ means $\sin(y) = x$

and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

So

$\sin y = x$

$\frac{d}{dx} \sin(y) = \frac{d}{dx} x$

$\cos(y)\, y' = 1 \Rightarrow y' = \frac{1}{\cos(y)}$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\cos(y) \ge 0$

so $\cos^2 y + \sin^2 y = 1 \Rightarrow \cos(y) = \sqrt{1-\sin^2(y)}$

$\Rightarrow y' = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

<u>Rule:</u> $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

Similarly you can derive

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$

<u>Ex:</u> Find the derivatives

(a) $\frac{d}{dx}\cos^{-1}(\sin^{-1}(x)) = f'(g(x))\cdot g'(x) = -\frac{1}{\sqrt{1-(\sin^{-1}(x))^2}}\cdot\frac{1}{\sqrt{1-x^2}}$

$f(y) = \cos^{-1}(y) \quad g(x) = \sin^{-1}(x)$

$f'(y) = -\frac{1}{\sqrt{1-y^2}} \quad g'(x) = \frac{1}{\sqrt{1-x^2}}$

(b) $\frac{d}{d\theta}\sin^{-1}(\sqrt{\sin\theta}) = \frac{d}{dy}\sin^{-1}(y)\Big|_{y=\sqrt{\sin\theta}} \cdot \frac{d}{d\theta}\sqrt{\sin\theta}$

$$= \frac{1}{\sqrt{1-\sin\theta}}\cdot\frac{1}{2\sqrt{\sin\theta}}\cdot\cos\theta$$

<u>Derivatives of logarithmic functions</u>

<u>Thm:</u> $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$

reason: $y = \log_a x \Leftrightarrow a^y = x$

$\frac{d}{dx} a^y = \frac{d}{dx} x$

$(\ln a) a^y y' = 1$

$\Rightarrow y' = \frac{1}{(\ln a) a^y} = \frac{1}{\ln(a) x}$.

So $\frac{d}{dx} \log_a x = \frac{1}{(\ln a) x}$.

In particular

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$$

Ex: Find $\frac{d}{dx} \sqrt{\ln x}$ if $\ln x > 0$.

$\frac{d}{dx} \sqrt{\ln x} = \frac{1}{2} \cdot (\ln x)^{-1/2} \frac{d}{dx} \ln x = \frac{1}{2x\sqrt{\ln x}}$

Ex: $\frac{d}{dx} \ln\left(\frac{x+1}{\sqrt{x-2}}\right) = \frac{d}{dx}\left[\ln(x+1) - \ln(x-2)^{1/2}\right] =$

$= \frac{d}{dx} \ln(x+1) - \frac{d}{dx} \ln(x-2)^{1/2} =$

$= \frac{1}{x+1} - \frac{1}{2} \frac{d}{dx} \ln(x-2) = \frac{1}{x+1} - \frac{1}{2(x-2)}$

Ex: $\frac{d}{dx} \ln(e^{-x} + x e^{-x}) =$

$= \frac{d}{dx} \ln(e^{-x}(1+x)) = \frac{d}{dx} \ln e^{-x} + \frac{d}{dx} \ln(x+1)$

$= \frac{d}{dx} -x + \frac{d}{dx} \ln(x+1) = -1 + \frac{1}{x+1}$

Ex $\frac{d}{dx} \ln(\ln(\ln x)) = \frac{1}{\ln(\ln x)} \cdot \frac{d}{dx} \ln(\ln x) =$

$= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{d}{dx} \ln x = \frac{1}{x(\ln x) \cdot \ln(\ln x)}.$

Logarithmic Differentiation

Ex Find $\frac{d}{dx} x^{\sqrt{x}}$ for $x > 0$

Steps in Logarithmic Differentiation:

1) Take natural logs of both sides of the equation $y = f(x)$ and use laws of logs to simplify.
2) Differentiate Implicitely
3) Solve for y'.

$$y = x^{\sqrt{x}}$$

$$\ln y = \ln\left(x^{\sqrt{x}}\right) = \sqrt{x}\,\ln x$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\sqrt{x}\,\ln x$$

$$\frac{1}{y}\cdot y' = \left(\frac{d}{dx}\sqrt{x}\right)\ln x + \sqrt{x}\,\frac{d}{dx}\ln x$$

$$\frac{y'}{y} = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}$$

$$\Rightarrow y' = \left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)x^{\sqrt{x}}$$

$$\Rightarrow \frac{d}{dx}x^{\sqrt{x}} = \frac{1}{\sqrt{x}}\left(1 + \frac{1}{2}\ln x\right)x^{\sqrt{x}}.$$

Ex: Differentiate $y = \sqrt{x}\,e^{x^2-x}(x+1)^{2/3}$ for $x > 0$

$$\ln y = \ln\left(\sqrt{x}\,e^{x^2-x}(x+1)^{2/3}\right) =$$

$$= \frac{1}{2}\ln x + x^2 - x + \frac{2}{3}\ln(x+1)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left[\frac{1}{2}\ln x + x^2 - x + \frac{2}{3}\ln(x+1)\right]$$

$$\frac{y'}{y} = \frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}$$

$$\Rightarrow y' = \sqrt{x}\,e^{x^2-x}(x+1)^{2/3}\left[\frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}\right]$$

Ex: Find y' if $x^y = y^x$

$\ln x^y = \ln y^x$

$y \ln x = x \ln y$

$\frac{d}{dx} y \ln x = \frac{d}{dx} x \ln y$

$y' \ln x + y \frac{1}{x} = 1 \cdot \ln y + x \frac{y'}{y}$

$y'\left(\ln x - \frac{x}{y}\right) = \ln y - \frac{y}{x}$

$y' = \dfrac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$

Ex: Find y' if $y = (\sin x)^{\ln x}$ where it is defined

$\ln y = \ln\left[(\sin(x))^{\ln x}\right] = \ln x \cdot \ln(\sin(x))$

$\frac{d}{dx} \ln y = \frac{d}{dx}\left[\ln x \cdot \ln(\sin(x))\right]$

$\frac{y'}{y} = \left[\frac{d}{dx} \ln x\right] \ln(\sin(x)) + \ln(x) \cdot \frac{d}{dx} \ln(\sin(x))$

$= \frac{\ln(\sin(x))}{x} + \ln(x) \cdot \frac{\cos x}{\sin x}$

$y' = (\sin x)^{\ln x}\left[\frac{\ln(\sin x)}{x} + \ln(x) \cot(x)\right]$