Lecture 15: Implicit Differentiation

So far we have considered derivatives of functions f and slopes of parametric curves $t \mapsto (x(t), y(t))$ that are given explicitly: you can evaluate in closed form $f(x)$ or $(x(t), y(t))$.

Sometimes functions are given implicitly.

Example: $x^2 + y^2 = 1$

There are two functions f such that $x^2 + (f(x))^2 = 1$, namely $f(x) = \sqrt{1-x^2}$ and $f(x) = -\sqrt{1-x^2}$ on the domain $[-1,1]$

Set $y = f(x)$. Then

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx} 1$$

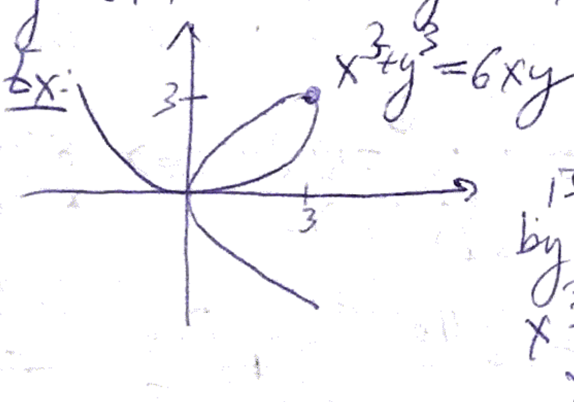
$$2x + \frac{d}{dx}(f(x))^2 = 0$$

$$\parallel$$

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

This formula is valid both for $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$.

Ex:



Difficult to solve for y

When we say that f is a function defined implicitly by $x^3+y^3=6xy$, we mean $x^3 + (f(x))^3 = 6x f(x)$ for all x in the domain of f.

(a) Find y', if $x^3 + y^3 = 6xy$

$$3x^2 + 3y^2 y' = 6\frac{d}{dx}(xy) = 6(y + xy') = 6y + 6xy'$$

$$3y^2y' - 6xy' = 6y - 3x^2$$

$$(3y^2 - 6x)y' = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$ provided $y^2 \neq 2x$

(b) Find the tangent at $(3,3)$

$$y'(3) = \frac{2(3) - 3^2}{3^2 - 2(3)} = \frac{-3}{3} = -1$$

So equation of tangent line is

$$y - 3 = -(x - 3). \iff x + y = 6$$

(c) At what point in the first quadrant is the tangent line horizontal?

$$y' = 0 \implies 6y - 3x^2 = 0 \text{ and } y^2 \neq 2x$$

$$y = \tfrac{1}{2}x^2$$

substituting back into $x^3 + y^3 = 6xy$ yields

$$x^3 + \left(\tfrac{1}{2}x^2\right)^3 = 6x\left(\tfrac{1}{2}x^2\right).$$

$$x^3 + \tfrac{1}{8}x^6 = 3x^3 \iff x^6 = 16x^3 \iff \cancel{x = 0} \text{ or } x^3 = 16$$

$$x^3(x^3 - 16) = 0$$

$$\iff x = (16)^{1/3}$$

Then $y = \frac{1}{2}(16)^{2/3}$ $\implies$ The point is $(16^{1/3}, \frac{1}{2}(16)^{2/3})$

Ex: Find $\frac{dy}{dx}$ for $e^{x/y} = x - y$

$$\frac{d}{dx} e^{x/y} = \frac{d}{dx}[x-y] = 1 - y'$$

$$\parallel$$

$$e^{x/y} \cdot \frac{d}{dx}\left[\frac{x}{y}\right] = e^{x/y} \frac{y\left[\frac{d}{dx}x\right] - x\frac{d}{dx}y}{y^2} = e^{x/y} \cdot \frac{y - xy'}{y^2}$$

$$\Leftrightarrow \quad y^2(1-y') = e^{x/y}(y - xy')$$

$$y^2 - y^2y' = ye^{x/y} - xy'e^{x/y}$$

$$y'(xe^{x/y} - y^2) = ye^{x/y} - y^2$$

$$\Rightarrow \quad y' = \frac{ye^{x/y} - y^2}{xe^{x/y} - y^2} \quad \text{as long as } xe^{x/y} - y^2 \neq 0$$

Ex: Find an equation of the tangent line to the curve $2(x^2+y^2)^2 = 25(x^2-y^2)$ at $(3,1)$

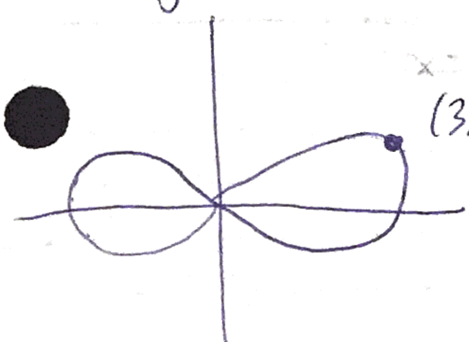
$$\frac{d}{dx}\left[2(x^2+y^2)^2\right] = \frac{d}{dx}\left[25(x^2-y^2)\right]$$

$$2 \cdot 2(x^2+y^2)\frac{d}{dx}(x^2+y^2) = 25(2x - 2yy')$$

$$4(x^2+y^2)(2x+2yy') = 25(2x-2yy')$$

Plug in $(3,1)$ $\Rightarrow$ $4(9+1)(2\cdot 3 + 2\cdot 1 \cdot y') = 25(2\cdot 3 - 2 \cdot 1 \cdot y')$

$$40(6+2y') = 25(6-2y')$$



$(3,1)$ $\Rightarrow$ $8(6+2y') = 5(6-2y')$

$$48 + 16y' = 30 - 10y'$$

$$26y' = -18$$

$$y' = -\frac{18}{26} = -\frac{9}{13}$$