Lecture 14: Derivatives and Parametric Curves (Cont'd)

Last time

If $t \mapsto (x(t), y(t))$ is a parametric curve, then the slope of the curve at time t is

$$\frac{dy}{dx}(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{x(t+h) - x(t)} = \frac{y'(t)}{x'(t)} = \frac{\frac{dy}{dt}(t)}{\frac{dx}{dt}(t)}$$

provided $x'(t) \neq 0$.

We say that the curve has a vertical tangent line at time t if $x'(t) = 0$ and $y'(t) \neq 0$.

Ex $x(t) = t^3 - 7t + 5$, $y(t) = 2t^3 - 3t^2 + 3t$

ⓐ Find the equation of the tangent line to the curve when $t = -1$.

$$\frac{dy}{dx}(t) = \frac{y'(t)}{x'(t)} = \frac{6t^2 - 6t + 3}{3t^2 - 7}$$

$$\Rightarrow \frac{dy}{dx}(-1) = -\frac{15}{4}$$

ⓑ Find all t when the tangent line has slope 3.

set $3 = \frac{dy}{dx}(t) = \frac{6t^2 - 6t + 3}{3t^2 - 7}$

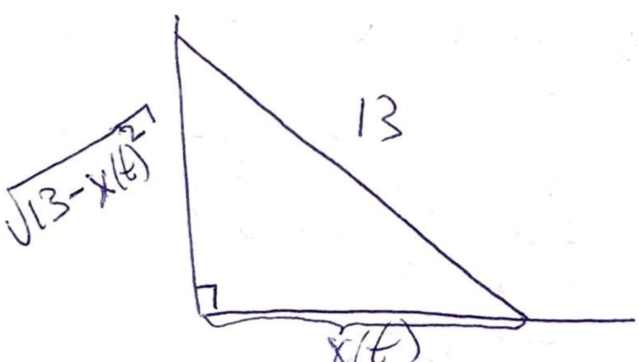
$\Rightarrow 9t^2 - 21 = 6t^2 - 6t + 3$

$3t^2 + 6t - 24 = 0 \Rightarrow t^2 + 2t - 8 = 0$

$(t+4)(t-2)$

$\Rightarrow t = 2, \ t = -4$.

Ex: A ladder 13 ft long rests against a wall. The bottom of the ladder slides away from the wall at a rate of 0.5 ft/sec. Consider the triangle formed by the ladder, the ground and the wall. At what rate is the area of the triangle changing when the bottom of the ladder is 12 ft from the wall?



$x(t) = 0.5t$

$$A(t) = \frac{1}{2} x(t) \cdot \sqrt{13 - x(t)^2} = \frac{1}{4} t \sqrt{13 - \frac{1}{4} t^2}$$

area at time t

$$= \frac{1}{4} \sqrt{13t^2 - \frac{1}{4} t^4}$$

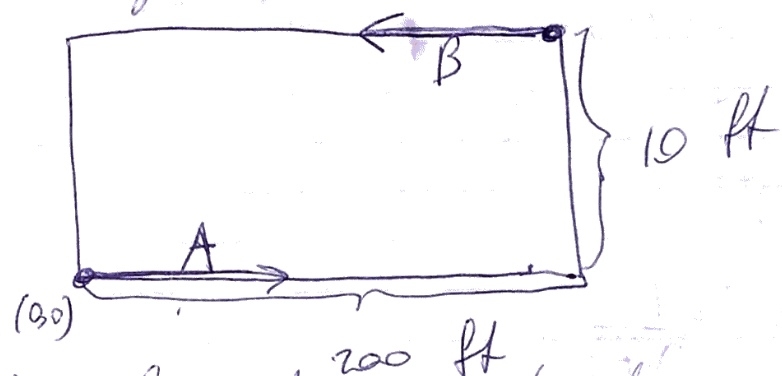
$$\Rightarrow \frac{dA}{dt}(t) = \frac{1}{4} \cdot \frac{1}{2} \cdot \left(13t^2 - \frac{1}{4} t^2\right)^{-1/2} \cdot \left(26t - t^3\right)$$

chain rule

$$\frac{dA}{dt}(24) = \frac{1}{4} \cdot \frac{1}{2} \cdot \left(13 \cdot (24)^2 - \frac{1}{4}(24)^2\right)^{-1/2} \left(26(24) - (24)^3\right)$$

[redacted]

Ex: Consider two ships, A and B, traveling in two lines



A travels at constant speed 15 ft/sec

B travels a constant speed 20 ft/sec

ⓐ After 2 seconds how fast is the distance between the two earls decreasing.

soln: Let $(x(t), y(t))$ be the position of A at time t

Let $(\hat{x}(t), \hat{y}(t))$ be the position of B at time t.

$\Rightarrow \quad y(t) = 0 \quad , \quad \hat{y}(t) = 10$

$x(t) = 15t \quad , \quad \hat{x}(t) = 200 - 20t$

Let (t) be the distance between A and B at time t.

$$\Rightarrow \quad D(t) = \sqrt{(y(t) - \hat{y}(t))^2 + (x(t) - \hat{x}(t))^2}$$

$$= \sqrt{100 + (15t + 20t - 200)^2}$$

$$= \sqrt{100 + (35t - 200)^2}$$

So $\dfrac{dD(t)}{dt} = \dfrac{1}{2} \cdot (100 + (35t - 200)^2)^{-1/2} \cdot (2(35t - 200)) \cdot 35.$

So $\dfrac{dD(2)}{dt} = \dfrac{1}{2} \cdot \dfrac{1}{70} \cdot (140)(35) = 35$