Lecture 13: Derivatives and Parametrized Curves

Chain Rule: If g is differentiable at x and f is differentiable at $g(x)$, then

$$(f \circ g)'(x) = f'(g(x))\,g'(x).$$

Finished last time with the consequence

$$\frac{d}{dx}[g(x)]^r = r[g(x)]^{r-1} g'(x)$$

Ex: Differentiate $h(x) = [x + (x + \sin^2(x))^3]^4$

$$h'(x) = 4[x + (x+\sin^2(x))^3]^3 \cdot \frac{d}{dx}[x + (x+\sin^2(x))^3]$$

$$= 4(x + (x+\sin^2(x))^3)^3 \cdot \left(1 + 3(x+\sin^2(x))^2 \frac{d}{dx}(x+\sin^2(x))\right)$$

$$= 4(x + (x+\sin^2(x)))^3 \cdot ((1 + 3(x+\sin^2(x))^2(1 + 2\sin(x)\cos x)$$

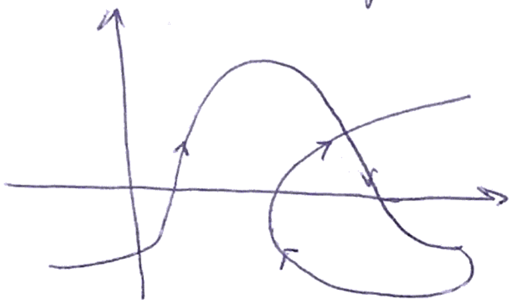
Ex: $\frac{d}{dx}\sqrt{1+xe^{-2x}} = \frac{1}{2}(1+xe^{-2x})^{-1/2} \cdot \frac{d}{dx}[1+xe^{-2x}] =$

$$= \frac{1}{2\sqrt{1+xe^{-2x}}}\left(x\frac{d}{dx}e^{-2x} + \left[\frac{d}{dx}x\right]e^{-2x}\right) =$$

$$= \frac{1}{2\sqrt{1+xe^{-2x}}}\left(-2xe^{-2x} + e^{-2x}\right) =$$

$$= \frac{e^{-2x}(1-2x)}{2\sqrt{1+xe^{-2x}}}$$

Consider a parametric curve $t \mapsto (x(t), y(t))$
The slope of the parametric curve at a point $(x(\bar{t}), y(\bar{t}))$ is defined to be



$$\lim_{t \to \bar{t}} \frac{y(t) - y(\bar{t})}{x(t) - x(\bar{t})}$$

and we denot it by $\frac{dy}{dx}(\bar{t})$

Notice
$$\frac{dy}{dx}(\bar{t}) = \lim_{t \to \bar{t}} \frac{y(t) - y(\bar{t})}{x(t) - x(\bar{t})} = \lim_{t \to \bar{t}} \frac{y(t) - y(\bar{t})}{t - \bar{t}} \cdot \frac{t - \bar{t}}{x(t) - x(\bar{t})} =$$
$$= \frac{y'(\bar{t})}{x'(\bar{t})}.$$

Rule: The slope of the parametric curve $t \mapsto (x(t), y(t))$ at a point $(x(\bar{t}), y(\bar{t}))$ is $\frac{dy}{dx}(\bar{t}) = \frac{y'(\bar{t})}{x'(\bar{t})}$. assuming $x'(\bar{t}) \neq 0$

Ex: A curve C is defined by the equation $t \mapsto (t^2, t^3 - 3t)$
ⓐ Show that C has two tangents at $(3, 0)$ and find their equations

step 1: Find all t such that $(t^2, t^3 - 3t) = (3, 0)$

$$\Rightarrow \left.\begin{array}{l} t^2 = 3 \\ t^3 - 3t = 0 \\ t(t^2 - 3) \end{array}\right\} \Rightarrow t = \sqrt{3} \text{ and } t = -\sqrt{3}.$$

Then the two tangents with slope

$$\frac{y'(\sqrt{3})}{x'(\sqrt{3})} = \left.\frac{3t^2-3}{2t}\right|_{t=\sqrt{3}} = \frac{3}{\sqrt{3}}$$

and

$$\frac{y'(-\sqrt{3})}{x'(-\sqrt{3})} = -\frac{3}{\sqrt{3}}.$$

So the two tangent lines are

$$y = \frac{3}{\sqrt{3}}(x-3) \text{ and } y = -\frac{3}{\sqrt{3}}(x-3)$$

(b) Find the points on C where the tangent line is horizontal or vertical.

Horizontal means: $\frac{y'(t)}{x'(t)} = 0 \Leftrightarrow y'(t) = 0$ and $x'(t) \neq 0$

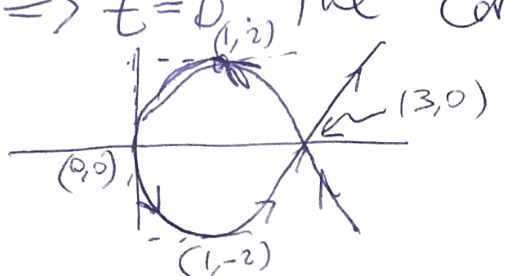
$0 = y'(t) = 3t^2 - 3 \Rightarrow t = \pm 1$

$x'(1) = 2$ and $x'(-1) = -2$

The corresponding points on C are $(1,-2)$ and $(1,2)$.

C has a vertical tangent when $x'(t) = 0$

$\Rightarrow t = 0$. The corresponding point is $(0,0)$



Ex. Find the points on the curves where the tangent is horizontal or vertical

a) $x = t^3 - 3t$, $y = t^3 - 3t$.

The curve traced out is the line $y = x$. So there are no horizontal or vertical tangents.

$$\frac{dy}{dx} = \frac{3t^2-3}{3t^2-3} = 1 \quad \text{for } t \neq \pm 1$$

b) $x = \cos\theta$ $\qquad y = \cos 3\theta$

$$\frac{dy}{dx} = \frac{-\sin(3\theta)(3)}{-\sin\theta} = \frac{3\sin(3\theta)}{\sin(\theta)}$$

Horizontal Tangent $\Leftrightarrow$ $3\sin(3\theta) = 0$ and $\sin(\theta) \neq 0$.

$3\theta = k \cdot \pi$
integer

$\theta = \frac{k \cdot \pi}{3}$ but also $\sin(\theta) \neq 0$
So k is not divisible by 3.

$\Rightarrow \theta = \pm\frac{\pi}{3}, \pm 2\cdot\frac{\pi}{3}, \pm 4\frac{\pi}{3}, \pm 5\frac{\pi}{3}, \pm 7\frac{\pi}{3} \ldots$

Vertical Tangent $\Leftrightarrow$ $\sin(\theta) = 0$ and $\sin(3\theta) \neq 0$.
$\Rightarrow \theta = \pi k$ but $\sin(3\theta) \neq 0$
integer

There are no vertical tangents.

(c) $X = e^{\sin\theta}, \quad y = e^{\cos\theta}$

$$\frac{dy}{dx} = \frac{e^{\cos(\theta)} \cdot (-\sin\theta)}{e^{\sin(\theta)} \cos\theta}$$

Horizontal Tangent $\Leftrightarrow$ $\sin\theta \cdot e^{\cos\theta} = 0$
and
$\cos\theta \, e^{\sin\theta} \neq 0$.

$\Leftrightarrow$ $\theta = k\pi$, k integer

Points are $\left(e^{\sin(k\pi)}, e^{\cos(k\pi)}\right)$ for k integer

i.e.

$(1, e)$ and $(1, 1/e)$

Vertical Tangent $\Leftrightarrow$ $e^{\sin(\theta)} \cos(\theta) = 0$ and $e^{\cos(\theta)} \sin(\theta) \neq 0$.

$\Leftrightarrow$ $\theta = k \cdot \frac{\pi}{2}$, k odd integer

Points are $\left(e^{\sin(k\frac{\pi}{2})}, e^{\cos(k\frac{\pi}{2})}\right)$ for k an odd integer

i.e. $(e, 0)$ and $(\frac{1}{e}, 0)$