Lecture 12: Midterm Review

Topics:

① Tangents to Circles
- If $l_1 \perp l_2$ (lines) with $l_1: y = m_1 x + b_1$, $l_2: y = m_2 x + b_2$,

then $m_1 m_2 = -1$

② Tangents and velocity
- velocity and average velocity

③ Limits
- limit laws, tricks for computing limits (combine fractions, multiply by conjugate)
- $\lim_{x \to \infty} f(x)$ or $\lim_{x \to -\infty} f(x)$

[Remember often best to divide num/den. by highest degree term in denominator]

- Squeeze Theorem.

If $f(x) \leq g(x) \leq h(x)$ and $L = \lim_{x \to a} f(x) = \lim_{x \to a} h(x)$,

then $\lim_{x \to a} g(x) = L$.

④ Continuity

If $\lim_{x \to a} f(x) = f(a)$, then f is continuous at a

Remember: f is continuous at a iff

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

- sums, products, quotients $\frac{f}{g}$ (with $g(a) \neq 0$), compositions of continuous functions are continuous

⑤ Derivative
- $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$
- Tangent line to $y = f(x)$ at a

[Know how to use the limit definition to compute the derivative]

- Differentiability $\Rightarrow$ Continuity.

- Higher Derivatives
- Derivative rules

$\frac{d}{dx} x^r = r x^{r-1}$, $\frac{d}{dx} e^x = e^x$

$(fg)'(x) =$ [redacted]$'(x) + f'(x)g(x)$

$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{[g(x)]^2}$

$(f+g)'(x) = f'(x) + g'(x)$

$(c\cdot f)'(x) = c f'(x)$.

Problems:

1) $\lim_{x\to 3} \frac{\sqrt{2x-1}}{x^2-6x+8} = \frac{\lim_{x\to 3}\sqrt{2x-1}}{\lim_{x\to 3} x^2-6x+8} = \frac{\sqrt{5}}{-1}$

(first "=": limit law; second "=": continuity)

2) $\lim_{x\to\infty}\left(\sqrt{4x^2-3x} - 2x\right) = \lim_{x\to\infty} \frac{(4x^2-3x) - 4x^2}{\sqrt{4x^2-3x} + 2x} =$

(multiply/divide by conjugate)

$= \lim_{x\to\infty} \frac{-3x}{2x + x\sqrt{4 - 3/x}} = \frac{-3}{2+2} = -\frac{3}{4}$

3) Is $H(x) = \begin{cases} (x-1)^2, & \text{if } x<0 \\ e^{x^2}, & \text{if } x \geq 0 \end{cases}$ continuous?

At $x>0$, H is continuous because it coincides with e^{x^2}. Similarly with $x<0$.

$\lim_{x\to 0^-} H(x) = \lim_{x\to 0^-} (x-1)^2 = 1$

$\lim_{x\to 0^+} H(x) = \lim_{x\to 0^+} e^{x^2} = 1$

So H is continuous at 0 also

4) Let $f(x) = x^2 - 7x + 3$

(a) Find equation of tangent line to $y = f(x)$ at $(1, -3)$

$f'(1) = 2(1) - 7 = -5$

So $y + 3 = -5(x - 1)$

(b) Find a point $(a, f(a))$ such that the tangent line to $y = f(x)$ at $(a, f(a))$ has y-intercept -1.

Tangent line at $(a, f(a))$

$y - (a^2 - 7a + 3) = (2a - 7)(x - a)$ ☆

Find a s.t. $(x, y) = (0, -1)$ satisfies the equation ☆

$-1 - (a^2 - 7a + 3) = (2a - 7)(-a)$

$= -2a^2 + 7a$

$a^2 - 4 = 0 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2.$

5) Determine if the function

$$f(x) = \begin{cases} x^2 - 3x + 3, & \text{if } x < 1 \\ \frac{1}{x} & \text{if } x > 1 \end{cases}$$

is differentiable at $x = 1$.

$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{(x^2 - 3x + 3) - 1}{x - 1} = \lim_{x \to 1^-} \frac{(x-2)(x-1)}{x-1} =$

$= \lim_{x \to 1^-} x - 2 = -1$

$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{\frac{1}{x} - 1}{x - 1} = \lim_{x \to 1^+} \frac{1 - x}{(x-1)x} = \lim_{x \to 1^+} -\frac{1}{x} = -1$

So $f'(1) = -1$

6) $\frac{d}{dx} \frac{x^2}{e^{-x}\sqrt[3]{x}} = \frac{d}{dx} \frac{x^{\overbrace{2-1/3}^{5/3}}}{e^{-x}} = x^{5/3} e^{x} = x^{5/3} e^{x} + \frac{5}{3} x^{2/3} e^{x}$

7) $\lim_{x\to -2^-} \frac{|x+2|}{x^2+7x+10} = \lim_{x\to -2^-} \frac{-(x+2)}{x^2+7x+10}$

$= \lim_{x\to -2^-} \frac{-(x+2)}{(x+2)(x+5)} = -\frac{1}{3}$

8) $\lim_{x\to +\infty} \frac{3x^{10}+x^8+3}{x^{10}-3x^6+2} = \lim_{x\to +\infty} \frac{3+x^{-2}+\frac{3}{x^{10}}}{1-3\frac{1}{x^4}+\frac{2}{x^{10}}} = 3.$

9) $\lim_{x\to 1^-} \frac{e^x+7}{(x-1)^3} = -\infty$

$\lim_{x\to 1^+} \frac{e^x+7}{(x-1)^3} = +\infty$

$\Rightarrow \lim_{x\to 1} \frac{e^x+7}{(x-1)^3}$ does not exist.

10) ⓐ If $f(x) = \begin{cases} x^2\sin(\frac{1}{x})+3, & \text{if } x\neq 0 \\ 1 & \text{if } x=0 \end{cases}$

continuous at 0?

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2\sin(\frac{1}{x})+3 = 3$

squeeze theorem

Notice: $-x^2 \leq x^2\sin(\frac{1}{x}) \leq x^2$

$\downarrow$ as $x\to 0$ $\quad$ $\downarrow$ as $x\to 0$

$0 \quad\quad 0$

So f is not continuous.

ⓑ Find a value c, if any, such that $f(x) = \begin{cases} x^2\sin(\frac{1}{x})+3, & \text{if } x\neq 0 \\ c & \text{if } x=0 \end{cases}$ is differentiable at 0.

Guess (by continuity) $c=3$.

$\Rightarrow \lim_{x\to 0} \frac{[x^2\sin(\frac{1}{x})+3]-3}{x-0} = \lim_{x\to 0} \frac{x^2\sin(\frac{1}{x})}{x} = \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$

Squeeze

11) (a)

$$\frac{d}{dx}\left(x^{\sqrt{2}}+\sqrt{2}\cdot x^{2}+\pi\right)=\sqrt{2}x^{\sqrt{2}-1}+2\sqrt{2}x$$

(b)

$$\frac{d}{dx}\,\frac{xe^{x}}{x+2}=\frac{(x+2)(e^{x}+xe^{x})-xe^{x}}{(x+2)^{2}}=$$

$$=\frac{xe^{x}+x^{2}e^{x}+2e^{x}+2xe^{x}-xe^{x}}{(x+2)^{2}}=$$

$$=\frac{x^{2}e^{x}+2e^{x}+2xe^{x}}{(x+2)^{2}}=\frac{e^{x}(x^{2}+2x+2)}{(x+2)^{2}}$$

(c)

$$\frac{d}{dx}\,\frac{\sqrt{x}}{e^{x}}=\frac{e^{x}\frac{1}{2\sqrt{x}}-\sqrt{x}\,e^{x}}{e^{2x}}=$$

$$=\frac{\frac{1}{2\sqrt{x}}-\sqrt{x}}{e^{x}}=\frac{1-2x}{2e^{x}\sqrt{x}}$$

— Differentiability ⟹ Continuity.