Lecture 11: The chain rule

Last time

Basic derivatives:

$$\frac{d}{dx} x^r = r x^{r-1}, \quad \frac{d}{dx} e^x = e^x, \quad \frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x.$$

New derivatives from old:

$$\frac{d}{dx} c \cdot f(x) = c \cdot \frac{d}{dx} f(x)$$

$$\frac{d}{dx} [f(x) + g(x)] = f'(x) + g'(x)$$

$$(f \cdot g)'(x) = f(x) g'(x) + f'(x) g(x)$$

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x) f'(x) - g'(x) f(x)}{g^2(x)}$$

Ex: $\frac{d}{dx} \underbrace{\sqrt{x^2+1}}_{= f(g(x))}$? or $\frac{d}{dx} \underbrace{e^{\sin(x)}}_{= f(g(x))}$?

where $f(y) = \sqrt{y}$, $g(x) = x^2 + 1$

where $f(y) = e^y$, $g(x) = \sin(x)$

Thm: If g is differentiable at x and f is differentiable at $g(x)$ then the function $F(x) = f(g(x))$ is differentiable at x with

$$F'(x) = f'(g(x)) \cdot g'(x)$$

[In other words
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$]

$\underline{Ex}$: $\frac{d}{dx}\sqrt{x^2+1} = \frac{d}{dx}(f\circ g)(x) = f'(g(x))g'(x) =$

$f'(y) = \frac{1}{2\sqrt{y}}$, $f(y) = \sqrt{y}$ $\quad = \frac{1}{2\sqrt{g(x)}} \cdot 2x =$

$g'(x) = 2x$, $g(x) = x^2+1$ $\quad = \frac{x}{\sqrt{x^2+1}}$

$\underline{Ex}$: Find $\frac{d}{dx} e^{\sin(x)}$

Set $f(y) = e^y$, $g(x) = \sin(x)$

$f'(y) = e^y$, $g'(x) = \cos(x)$

$\Rightarrow \frac{d}{dx} e^{\sin(x)} = \frac{d}{dx}(f\circ g)(x) = f'(g(x))\, g'(x) =$

$= e^{g(x)} \cdot \cos(x) = e^{\sin(x)} \cos(x)$

$\underline{Ex}$: Find $\frac{d}{dx} \sin(x^2)$ and $\frac{d}{dx} \sin^2(x)$

Set $f(y) = \sin(y)$ and $g(x) = x^2$

$f'(y) = \cos(y)$ $\quad g'(x) = 2x$

$\Rightarrow \frac{d}{dx} \sin(x^2) = \frac{d}{dx}(f\circ g)(x) = f'(g(x)) \cdot g'(x) = \cos(x^2) \cdot (2x)$

Set $f(y) = y^2$ and $g(x) = \sin(x)$

$f'(y) = 2y$ $\quad g'(x) = \cos(x)$

$\Rightarrow \frac{d}{dx} \sin^2(x) = \frac{d}{dx}(f\circ g)(x) = f'(g(x)) \cdot g'(x) =$

$= 2\sin(x)\cos(x) = \sin(2x)$

"proof" of the chain rule:

$$(f\circ g)'(x) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} =$$

$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

$$= f'(g(x)) \cdot g'(x)$$

[Remark: The proof is not completely correct because $g(x+h) - g(x)$ could be zero. See the book for a more delicate argument]

Ex: $\frac{d}{dx}[g(x)]^r = ?$

Set $f(y) = y^r \Rightarrow f'(y) = ry^{r-1}$

Then

$$\frac{d}{dx}(g(x))^r = \frac{d}{dx} f(g(x)) = f'(g(x))g'(x) =$$

$$= r[g(x)]^{r-1} \cdot g'(x).$$

Useful Rule:

$$\frac{d}{dx}[g(x)]^r = r[g(x)]^{r-1} g'(x).$$

Ex: Differentiate $y = e^{\sqrt{x}}$

$f(y) = e^{y}$ $\quad g(x) = \sqrt{x}$

$f'(y) = e^{y}$ $\quad g'(x) = \dfrac{1}{2\sqrt{x}}$

So $\dfrac{d}{dx} e^{\sqrt{x}} = f'(\sqrt{x}) \cdot g'(x) = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

Ex: $\dfrac{d}{dx} \underbrace{(2x^3+5)}_{g(x)}{}^4 = 4(2x^3+5)^3 \cdot \dfrac{d}{dx}[2x^3+5] =$

$= 4(2x^3+5)^3(6x)$

Ex: Find $\dfrac{d}{dx} \cos^2(\sin x)$

$\dfrac{d}{dx} \cos^2(\sin(x)) = \dfrac{d}{dx}[\cos(\sin(x))]^2 =$

$= 2\cos(\sin(x)) \cdot \dfrac{d}{dx} \cos(\sin(x))$

Set $f(y) = \cos(y)$, $\quad g(x) = \sin(x)$

$f'(y) = -\sin(y)$, $\quad g'(x) = \cos(x)$

$= 2\cos(\sin(x)) \cdot f'(\sin(x)) \cdot g'(x) =$

$= 2\cos(\sin(x))(-\sin(\sin(x)))\cos(x)$

$= -\sin(2\sin(x))\cos(x)$

Ex:

$$\frac{d}{dx} e^{\sec(3x)} = e^{\sec(3x)} \frac{d}{dx} \sec(3x) =$$

$$= e^{\sec(3x)} \cdot \sec(3x) \tan(3x) \frac{d}{dx}(3x) =$$

$$= 3e^{\sec(3x)} \sec(3x) \tan(3x)$$

Next, let's compute $\frac{d}{dx} a^x$.

$$\frac{d}{dx} a^x = \frac{d}{dx} \Big[\underbrace{e^{\ln(a)}}_{= a}\Big]^x = \frac{d}{dx} e^{x \cdot \ln(a)}$$

Remember $a = e^{\ln(a)} \Leftrightarrow \ln(a) = \ln\left[e^{\ln(a)}\right] =$

$$= \ln(a) \cdot \ln e = \ln(a)$$

So

$$= e^{x \cdot \ln(a)} \cdot \frac{d}{dx} x \ln(a) = e^{x \ln(a)} \ln(a)$$

$$= \left[e^{\ln(a)}\right]^x \ln(a) = a^x \ln(a).$$

Then:

$$\boxed{\frac{d}{dx} a^x = a^x \ln(a)}$$