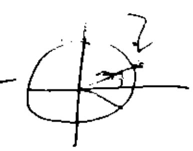
Lecture 10: Derivatives of Trigonometric Functions

HW Make sure to read "Sinusoidal functions supplement" on the course webpage:

www.math.washington.edu/~m124

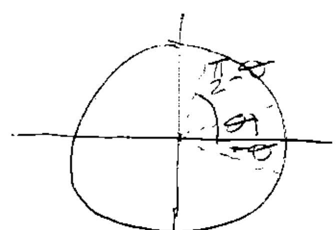
Useful Trig Identities:

$\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$, $\sin(-\theta) = -\sin\theta$



$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\cos(-\theta) = \cos\theta$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$



$\cos^2\theta + \sin^2\theta = 1$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

More useful identities on page 2 of the book

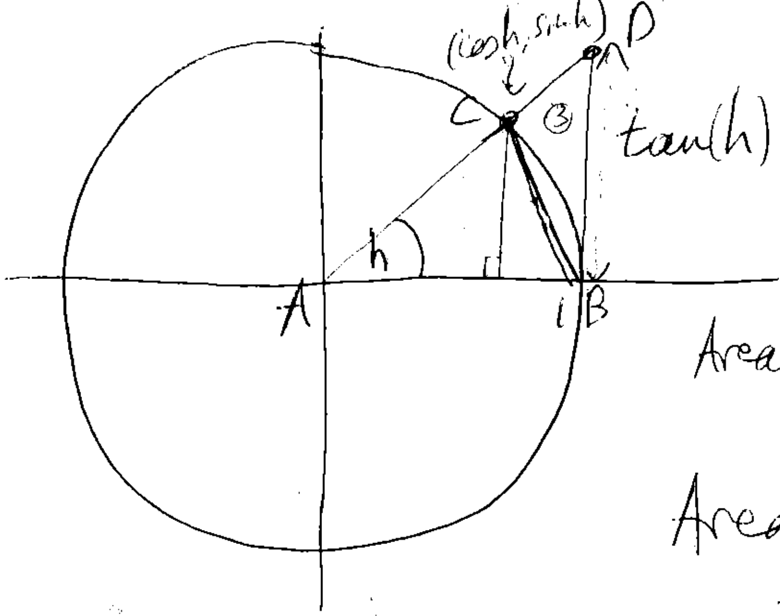
Let's try to compute $\frac{d}{dx}\sin x$

$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{[\sin(x)\cos(h) + \cos(x)\sin(h)] - \sin(x)}{h}$$

$$= \lim_{h\to 0} \sin(x)\cdot\frac{\cos(h)-1}{h} + \lim_{h\to 0}\cos(x)\frac{\sin(h)}{h}$$

$$= \sin(x)\left[\lim_{h\to 0}\frac{\cos(h)-1}{h}\right] + \cos(x)\left[\lim_{h\to 0}\frac{\sin(h)}{h}\right]$$

<u>Goal</u>: Compute $\lim_{h \to 0} \frac{\sin(h)}{h}$ and $\lim_{h \to 0} \frac{\cos(h)-1}{h}$



Area of triangle ABC

$$= \frac{1}{2}(1)\cdot \sin(h) = \frac{\sin(h)}{2}$$

Area of Wedge spanne

$$= \frac{h}{2\pi}\cdot \pi = \frac{h}{2}$$

Area of triangle ABD

$$= \frac{1}{2}(1)\tan(h) = \frac{\tan(h)}{2}$$

$$\Rightarrow \frac{\sin(h)}{2} \le \frac{h}{2} \le \frac{\tan(h)}{2}$$

Similarly

$$\frac{|\sin(h)|}{2} \le \frac{|h|}{2} \le \frac{|\tan(h)|}{2}$$

$$1 \le \frac{|h|}{|\sin h|} \le \frac{|\tan(h)|}{|\sin(h)|} = \frac{1}{|\cos(h)|}$$

$$\Rightarrow 1 \ge \frac{|\sin(h)|}{|h|} \ge |\cos(h)|$$

For $h \to 0$, $\frac{|\sin(h)|}{|h|} = \frac{\sin(h)}{h}$ and $|\cos(h)| = \cos(h)$

so $1 \ge \frac{\sin(h)}{h} \ge \cos(h)$

So $\boxed{\lim_{h \to 0} \frac{\sin h}{h} = 1}$

Next, let's compute

$$\lim_{h \to 0} \frac{\cos(h)-1}{h} = \lim_{h \to 0} \frac{(\cos(h)-1)(\cos(h)+1)}{h(\cos(h)+1)} =$$

$$= \lim_{h \to 0} \frac{\cos^2(h)-1}{h(\cos(h)+1)} =$$

$$= \left[\lim_{h \to 0} \frac{\sin h}{h}\right] \cdot \left[\lim_{h \to 0} \frac{-\sin(h)}{\cos(h)+1}\right]$$

$$= 1 \cdot 0 = 0$$

So $\boxed{\lim_{h \to 0} \frac{\cos(h)-1}{h} = 0}$

Finally

$$\frac{d}{dx} \sin(\cancel{x}) = \sin(x) \cdot 0 + \cos(x) \cdot (1) = \cos(x)$$

<u>Thm</u>: $\frac{d}{dx} \sin(x) = \cos(x)$

Similar argument shows

$$\frac{d}{dx} \cos(x) = -\sin(x)$$

$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$ computed by the quotient rule

Ex $\frac{d}{dx} \sec(x) = \frac{d}{dx} \frac{1}{\cos(x)} = \frac{\cos(x)\cdot\frac{d}{dx}1 - 1\cdot\frac{d}{dx}\cos x}{\cos^2(x)} =$

$= +\frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x \cdot \sec x$

<u>Derivaties of Trig Functions:</u>

$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\csc x = -\csc(x)\cot(x)$

$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec(x)\tan(x)$

$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2(x)$

Ex $\frac{d}{dx} e^x\cos x = e^x\frac{d}{dx}\cos x + \left[\frac{d}{dx}e^x\right]\cos x =$

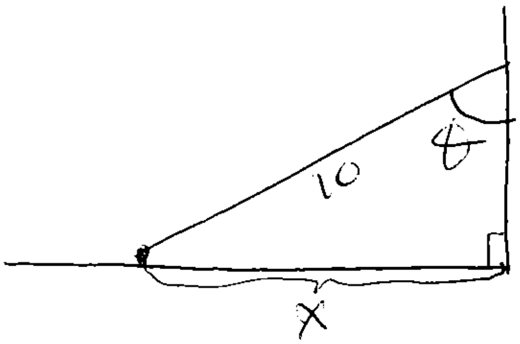
$= -e^x\sin x + e^x\cos x$

$= e^x[\cos x - \sin x]$

Find: ~~~~ Set $f(x) = \sin(x)$. Compute $f^{(99)}(x)$.

$$\left.\begin{aligned} f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f^{(3)}(x) &= -\cos(x) \\ f^{(4)}(x) &= \sin(x) \end{aligned}\right\} \Rightarrow \begin{aligned} f^{(96)}(x) &= \sin(x) \\ f^{(97)}(x) &= \cos(x) \\ f^{(98)}(x) &= -\sin(x) \\ f^{(99)}(x) &= -\cos(x) \end{aligned}$$

Ex: A ladder 10 ft long rests against a vertical wall. Let θ be the angle between top of the ladder and the wall



If the bottom of the ladder slides away from the wall, how fast does the distance from the bottom of the ladder to the wall change with respect to θ when $\theta = \frac{\pi}{3}$?

$$x = f(\theta) = 10\sin\theta$$

$$\text{So } f'(\theta) = 10\cos\theta \Rightarrow f'\left(\frac{\pi}{3}\right) = \cancel{10\cdot\frac{\sqrt{3}}{2} = 5\sqrt{3}}$$

$$= 10\cdot\frac{1}{2} = 5$$

Ex: Evaluate $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta + \tan\theta}$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta + \tan\theta} = \lim_{\theta\to 0} \frac{\frac{\sin\theta}{\theta}}{1 + \frac{\frac{\sin\theta}{\theta}}{\cos\theta}} = \frac{\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}}{1 + \frac{\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}}{\lim\limits_{\theta\to 0}\cos\theta}} = \frac{1}{2}.$$