Lecture 6: Subexponential RVs

Ex: Let $Z \sim N(0,1)$. Let's compute

$$\mathbb{E}\left[e^{\lambda(Z^2-1)}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\lambda(x^2-1)} e^{-x^2/2}\, dx$$

$$= \frac{e^{-\lambda}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(1-2\lambda)x^2/2}\, dx$$

$$= \begin{cases} \dfrac{e^{-\lambda}}{\sqrt{1-2\lambda}} & \text{if } \lambda \le \frac{1}{2} \\ +\infty, & \text{if } \lambda > \frac{1}{2} \end{cases}$$

Defn: X with mean $\mu = \mathbb{E}X$ is <u>subexponential</u> with <u>parameters</u> (σ, α) if

$$\mathbb{E}\left[e^{\lambda(X-\mu)}\right] \le e^{\frac{\sigma^2\lambda^2}{2}} \quad \forall\, |\lambda| \le \frac{1}{\alpha}$$

Back to example $Z \sim N(0,1)$

$$\mathbb{E}\left[e^{\lambda(Z^2-1)}\right] \le \frac{e^{-\lambda}}{\sqrt{1-2\lambda}} \le e^{\frac{4\lambda^2}{2}} \quad \forall\, |\lambda| < \frac{1}{4}$$

So Z^2 is $(2,4)$-subexponential.

Thm (Subexponential tail bound)

Let X be subexponential with (ν, α). Then

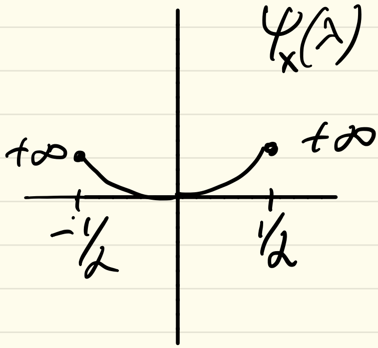
$$\mathbb{P}[X-\mu \geq t] \leq \begin{cases} e^{-\frac{t^2}{2\nu^2}}, & \text{if } |t| \leq \frac{\nu^2}{\alpha} \\ e^{-\frac{t}{\alpha}+\frac{\nu^2}{2\alpha^2}}, & \text{o.w.} \end{cases}$$

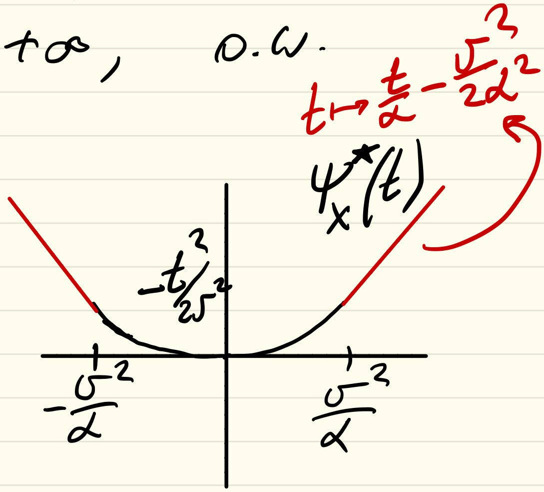
pf: Back to Chernoff

$$\log \mathbb{P}[X-\mu \geq t] \leq -\psi_X^*(t)$$

where

$$\psi_X(\lambda) = \log \mathbb{E} e^{\lambda(X-\mu)} = \begin{cases} \frac{\nu^2\lambda^2}{2}, & \text{if } |\lambda| \leq \frac{1}{\alpha} \\ +\infty, & \text{o.w.} \end{cases}$$





Thm: (Bernstein) Let X be subexponential with parameter (σ, α) and mean $\mu = \mathbb{E}X$. Then

$$\mathbb{P}\left[|X-\mu| \geq t\right] \leq 2\exp\left[-\left(\frac{t^2}{\sigma^2} \wedge \frac{t}{\alpha}\right)/2\right]$$

minimum

Pf:

$$t \mapsto \left(\frac{t^2}{\sigma^2} \wedge \frac{t}{\alpha}\right)/2$$

Lemma: [Sum Rule]

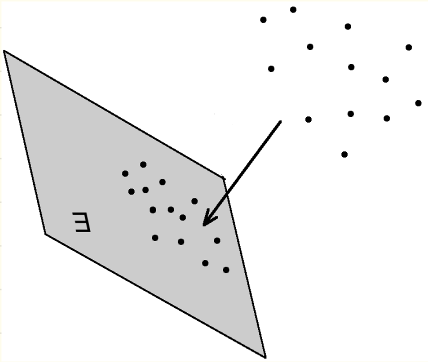
$$\left.\begin{array}{l} X_i \text{ are } (\sigma_i, \alpha_i)\text{-subexp} \\ \text{independent } \forall i = 1, \ldots, n \end{array}\right\} \Longrightarrow \sum_{i=1}^{n} X_i \text{ is } (\|\sigma\|_2, \|\alpha\|_\infty)\text{-subexp}$$

Thm (Bernstein for sums) Let $X_1, \ldots, X_n$ be independent sub-exponential with parameters (σ_i, α_i), and with mean $\mu_i = \mathbb{E}X_i$. Then

$$\mathbb{P}\left[\left|\sum_{i=1}^{N}(X_i - \mu_i)\right| > t\right] \leq 2\exp\left[-\frac{1}{2}\left(\frac{t^2}{\|\sigma\|_2^2} \wedge \frac{t}{\|\alpha\|_\infty}\right)\right]$$

Application: Dimensionality Reduction

Given $u_1, \ldots, u_m \in \mathbb{R}^d$ with $m \ll d$, can one map $u_1, \ldots, u_m$ into a lower dimensional space with low distortion?



Thm: (Johnson-Lindenstrauss)
Fix $\varepsilon, \delta \in (0,1)$, a set $U \subseteq \mathbb{R}^d$ of m points and a number $n > \frac{16 \ln(\frac{m^2}{\delta})}{\varepsilon^2}$.
Let $X \in \mathbb{R}^{n \times d}$ consist of i.i.d $N(0,1)$ entries.
Then with probability $1-\delta$, the map $f(u) = \frac{1}{\sqrt{n}} X u$ satisfies

$$1 - \varepsilon \leq \frac{\|f(u) - f(v)\|_2^2}{\|u - v\|_2^2} \leq 1 + \varepsilon \quad \forall u, v \in U.$$

pf: Observe

$$\frac{\|Xu\|_2^2}{\|u\|_2^2} = \sum_{i=1}^{n} \underbrace{\left\langle x_i, \frac{u}{\|u\|} \right\rangle}_{\text{i.i.d } N(0,1)}^2$$

x_i: i'th row of X

$\Rightarrow \frac{\|Xu\|_2^2}{\|u\|_2^2}$ is $(2\sqrt{n}, 4)$ – subexponential

$\Rightarrow$ Bernstein:

$$\mathbb{P}\left[\left|\frac{\|Xu\|_2^2}{n\|u\|_2^2} - 1\right| > \varepsilon\right] \leq 2\exp\left[-\left(\frac{n\varepsilon^2}{8} \wedge \frac{n\varepsilon}{8}\right)\right] = 2\exp\left(-\frac{n\varepsilon^2}{8}\right) \ \forall_{0 \leq \varepsilon \leq 1}$$

So for any i,j get

$$\mathbb{P}\left[\frac{\|f(u_i - u_j)\|^2}{\|u_i - u_j\|_2^2} \notin [1-\varepsilon, 1+\varepsilon]\right] \leq 2e^{-n\varepsilon^2/8}$$

Take union bound over $\binom{m}{2}$ pairs of points

$$2\binom{m}{2}e^{-n\varepsilon^2/8} \leq m^2 e^{-n\varepsilon^2/8} = \delta \quad \blacksquare$$

<u>Question</u>: What if $m = \infty$ but $\mathcal{U}$ has few "degrees of freedom"?