Lecture 4 Hoeffding & Chernoff Inequalities

Lemma: (Sum Rule) If X_i are independent and σ_i-subGaussian, then $\sum_{i=1}^n X_i$ is $\underbrace{\sqrt{\sum_i \sigma_i^2}}_{\|\sigma\|_2}$ - subGaussian.

pf: $\mathbb{E}\exp(\lambda \sum X_i) \underset{\text{independence}}{=} \prod \mathbb{E}\exp(\lambda X_i) \le \prod \exp(\lambda^2 \sigma_i^2 / 2)$

$= \exp\left(\frac{\lambda^2}{2} \sum \sigma_i^2\right)$ ☒

Cor: (Hoeffding)

Suppose $X_1, \dots, X_n$ are independent with $\mathbb{E}X_i = \mu_i$ and are σ_i-sub-Gaussian. Then

$$\mathbb{P}\left[\sum_{i=1}^n (X_i - \mu_i) \ge t\|\sigma\|_2\right] \le \exp\left\{-\frac{t^2}{2}\right\}$$

$\Longrightarrow$ If $\mu_i = \mu$, $\sigma_i = \sigma$, then

$$P\left[\sum_{i=1}^n (X_i - \mu) \ge t\sigma\sqrt{n}\right] \le \exp\left\{-\frac{t^2}{2}\right\}$$

<u>Ex:</u> (Coin flipping)

Let $X_1, \dots, X_n$ be Bernoulli$(\frac{1}{2})$ and set $S_n = \sum_{i=1}^n X_i$. Then

σ_i

$$P\left(S_n \geq \frac{3}{4}n\right) = P\left(\sum_{i=1}^n (X_i - \mathbb{E}X_i) \geq \frac{1}{4}n\right)$$

$$\underset{\text{Hoeffding}}{\leq} \exp\left(-\frac{n}{2}\right)$$

Hoeffding's inequality implies confidence bounds for mean estimation:

if X_i are iid and σ_i-subGauss, then $\hat{x} = \frac{1}{n}\sum_{i=1}^n x_i$ satisfies

$$P\left(|\hat{x} - \mathbb{E}x_1| \leq \sqrt{\frac{2\sigma^2 \log(1/p)}{n}}\right) \geq 1 - p$$

Can one achieve a similar guarantee without subGaussian assumption with a different estimator $\hat{x}$?

<u>Answer</u>: yes, almost!

Thm: (Median of means)

Consider $X \in \mathbb{R}$ with $\mathbb{E}X = \mu$ and $Var(X) = \sigma^2$. Let $X_1, \dots, X_n$ be i.i.d realizations of X. Subdivide into $k = 18\log(\frac{1}{p})$ bins and form the empirical means $\hat{x}_j$ for $j = 1, \dots, k$.

Then $\hat{x} = median(\hat{x}_1, \dots, \hat{x}_k)$ satisfies

$$\mathbb{P}\left[|\hat{x} - \mu| \le \sqrt{\frac{54\sigma^2 \log(\frac{1}{p})}{n}} \right] \ge 1 - p$$

pf: By Chebychev

$$\mathbb{P}\left[|\hat{X}_i - \mu| \ge \sqrt{\frac{3\sigma^2 k}{n}} \right] \le \frac{\frac{\sigma^2}{n/k}}{\frac{3\sigma^2 k}{n}} = \frac{1}{3} \quad \forall i.$$

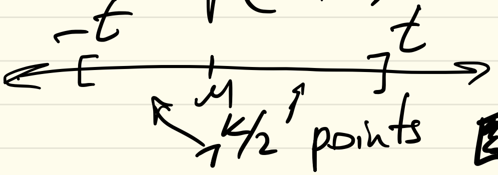
Let $\mathbb{1}_i$ be indicator of this event

Then by Hoeffding,

$$\mathbb{P}\left[\frac{1}{k}\sum_{i=1}^{k} \mathbb{1}_i > \frac{1}{2} \right] \ge 1 - \exp\left(-\frac{k}{18}\right)$$

in this event,

$\Rightarrow |\hat{x} - \mu| \le \sqrt{\frac{3\sigma^2 k}{2n}}$



$\blacksquare$

Notice that in contrast to sub-Gaussian case, $\hat{x}$ depends on confidence level p.