Chapter 2

Random Vectors in High Dimensions

- Concentration of the norm
- Isotropy
- Similarity of Normal and Spherical
- Sub-Gaussian and Subexponential random vectors.

Two man results we'll prove:

1) Sub-Gaussian vectors concentrate around a sphere.
2) Two independent isotropic subGaussian random vectors are nearly orthogonal in high dimensions.

We next investigate the behavior of random vectors in high dimensions

Concentration of the norm

Let $X = (X_1, \dots, X_d) \in \mathbb{R}^d$ have independent σ-subGaussian coordinates with

$$\mathbb{E} X_i = 0 \quad \text{and} \quad \mathbb{E} X_i^2 = 1$$

What can we expect from

$$\|X\|_2^2 \quad \text{and} \quad \|X\|_2$$

Lemma: Suppose y is σ-subGaussian. Then y^2 is $(\sigma, 4\sigma^2)$ subexponential

pf sketch:

Step 1: Estimate $\mathbb{E}[|y|^r] \leq r\, 2^{r/2} \sigma^r \Gamma\left(\frac{r}{2}\right)$

using $\mathbb{E}[|y|^r] = \int_0^\infty \mathbb{P}[|y| > t^{1/r}]\, dr$

Step 2: Use Taylor Expansion

$$\mathbb{E}\left[e^{\lambda(y^2 - \mathbb{E}y^2)}\right] \leq 1 + \sum_{r=2}^{\infty} \lambda^r 2^{r+1} \sigma^{2r} \leq 1 + \frac{8\lambda^2\sigma^4}{1 - 2\lambda\sigma^2}$$

$$\leq \exp(\dots) \quad \square$$

Cor: Let $X = (X_1, \dots, X_d) \in \mathbb{R}^d$ have independent σ-subGaussian coordinates with

$$\mathbb{E} X_i = 0 \quad \text{and} \quad \mathbb{E} X_i^2 = 1$$

Then $$\mathbb{P}\left[\left| \|X\|_2^2 - d \right| \geq td \right] \leq 2\exp\left(-\frac{d}{4\sigma^2}(t \wedge t^2) \right)$$

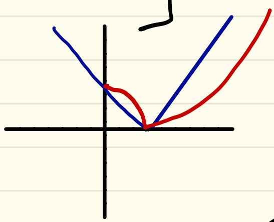
$$\mathbb{P}\left[\left| \|X\|_2 - \sqrt{d} \right| \geq t\sqrt{d} \right] \leq 2\exp\left(\frac{-dt^2}{4\sigma^2} \right)$$

pf: $$\|X\|_2^2 = \sum_{i=1}^d X_i^2 \Rightarrow \begin{cases} \mathbb{E}\|X\|_2^2 = d \\ \|X\|_2^2 \text{ is } (\sigma\sqrt{d}, 4\sigma^2) \\ \quad \text{subexponential.} \end{cases}$$

Bernstein $\Rightarrow$

$$\mathbb{P}\left[\left| \frac{1}{d}\|X\|_2^2 - 1 \right| \geq t \right] \leq 2\exp\left[-\frac{d}{4\sigma^2}(t \wedge t^2) \right]$$

Observation: for any $z \geq 0$, have



$$|z-1| \geq t \Rightarrow |z^2 - 1| \geq t \vee t^2.$$

So $$\mathbb{P}\left[\left| \frac{1}{\sqrt{d}}\|X\|_2 - 1 \right| \geq t \right] \leq \mathbb{P}\left[\left| \frac{1}{d}\|X\|_2^2 - 1 \right| \geq t \vee t^2 \right]$$
$$\leq 2\exp\left(\frac{-dt^2}{4\sigma^2} \right) \qquad \boxtimes$$

Isotropic Vectors

Recall for $X \in \mathbb{R}^d$, covariance

$$\mathrm{cov}(X) = \mathbb{E}\left((X-\mu)(X-\mu)^T\right)$$

where $\mu = \mathbb{E}X$.

Defn: A random vector $X \in \mathbb{R}^d$ with $\mathbb{E}X = 0$ is isotropic if

$$\Sigma(X) := \mathbb{E}XX^T = I_d$$

Remark: If $\Sigma = \Sigma(X)$ is invertible, then

$$Z := \Sigma^{-1/2}(X-\mu) \text{ is isotropic.}$$

Lemma: X is isotropic iff

$$\mathbb{E}\langle X, y\rangle^2 = \|y\|_2^2 \quad \forall y \in \mathbb{R}^d$$

pf: X is isotropic iff $\mathbb{E}XX^T = I$

iff $y^T \mathbb{E}XX^T y = y^T y$

iff $\mathbb{E}y^T X X^T y = \|y\|_2^2$

iff $\mathbb{E}\langle X, y\rangle^2 = \|y\|_2^2$ $\boxtimes$

Thus if $\mathbb{E}X=0$, then X isotropic iff marginal $\langle X, \frac{y}{\|y\|}\rangle$ has unit variance $\forall_y \in \mathbb{R}^d$

Lemma: Let $X \in \mathbb{R}^d$ be isotropic. Then

$$\mathbb{E}\|X\|_2^2 = d.$$

Moreover, if X and Y are two independent isotropic vectors, then

$$\mathbb{E}\langle X, Y\rangle^2 = d$$

pf: First

$$\|X\|_2^2 = X^T X = \operatorname{trace}(XX^T)$$

$$\Rightarrow \mathbb{E}\|X\|_2^2 = \operatorname{trace}(I_d) = d.$$

Next $\mathbb{E}\langle X, Y\rangle^2 = \mathbb{E}_Y\left[\mathbb{E}_X\langle X, Y\rangle^2 \mid Y\right]$

$$= \mathbb{E}_Y \|Y\|_2^2 = d$$

$\square$

Let X and y be independent and isotropic

Then $\|X\| \sim \sqrt{d}$ and $\left\langle \frac{X}{\|X\|}, \frac{y}{\|y\|} \right\rangle \sim \frac{1}{\sqrt{d}}$.

$\|y\| \sim \sqrt{d}$

$\therefore$ Almost orthogonal.

Can be made rigorous by assuming light tails.

Examples of isotropic RV:

1) Spherical $X \sim \mathrm{Unif}(\sqrt{d}\, S^{d-1})$
2) Symmetric Bernoulli: $X \sim \mathrm{Unif}(\{-1,1\}^d)$
3) Any vector $X = (X_1, \dots, X_d)$, where X_i are independent, zero mean, unit variance.
4) Coordinate $\mathrm{Unif}(\{\sqrt{d}\, e_i\}_{i=1}^d)$
5) Gaussian $g = (g_1, \dots, g_d) \sim N(0, I_d)$
Recall this means g_i are i.i.d. $N(0,1)$

$\Rightarrow$ Density
$$p(x) = \prod_{i=1}^d p_i(x) = \prod_{i=1}^d \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\|x\|^2/2}.$$

$\Rightarrow N(0, I_d)$ is rotation invariant.

Exercise:
Let $g \sim N(0, I_d)$. Then
$$r := \|g\|_2 \quad \text{and} \quad \theta = \frac{g}{\|g\|_2}$$
are independent random variables and
$$\theta \sim \mathrm{Unif}(S^{d-1})$$

Defn: X in $\mathbb{R}^d$ is σ-subGaussian if $\langle X, u\rangle$ is σ-subGaussian $\forall u \in S^{d-1}$

Ex: Let $X = (X_1, \ldots, X_d)$ be RV with independent σ-subGaussian X_i.
Then X is σ-subGaussian.

Ex: 1) $N(0, I_d)$ is 1-subGaussian
2) $Unif(\{-1,1\}^d)$ is 1-subGaussian
3) $Unif(\{\sqrt{d}\, e_i\}_{i=1}^d)$ is σ-subGaussian with $\sigma \simeq \sqrt{\frac{d}{\log(d)}}$

Way too big to be useful

4) $Unif(\sqrt{d}\, S^{d-1})$ is c-subGaussian for a constant c.