$\boxed{\text{Chapter 8}}$

<u>Sparse Recovery</u>

- High Dimensional Signal Recovery
- signal recovery based on M^* bound
- exact recovery based on escape
- deterministic conditions for recovery
- Recovery with noise (LASSO/BPDN)

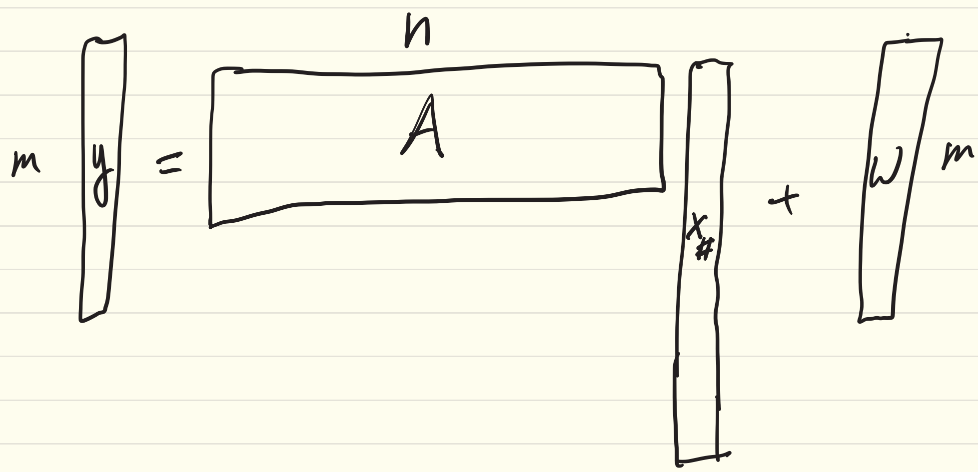
Problem: Fix a vector $x_\# \in \mathbb{R}^n$ and we get to see $y \in \mathbb{R}^m$ satisfying

$$y = Ax_\# + w$$

where $w \in \mathbb{R}^m$ is an error vector.

Notation

- $x_\#$ is called the signal
- y is called the measurement
- A is called the design matrix
- w is called the noise



We'll be interested in setting $m \ll n$

under additional prior information $x_\# \in T$

For the time being assume $w=0$.

Summary:

Having access to y, find $x \in \mathbb{R}$ s.t.

$\circledast$ $y = Ax$ and $x \in T$.

Thm: Suppose $A \in \mathbb{R}^{m \times n}$ has independent, isotropic, σ-subGaussian rows. Then any point $\hat{x}$ satisfying $\circledast$ satisfies

$$\mathbb{E}\|\hat{x} - x_\#\|_2 \leq \frac{C\sigma^2 \omega(T)}{\sqrt{m}}$$

pf: Since $\hat{x}, x_\# \in T$ satisfy $A\hat{x} = Ax_\# = y$, we know $\hat{x} \in x_\# + \ker A$. So M^* bound

$$\Rightarrow \mathbb{E}\|\hat{x} - x_\#\|_2 \leq \mathbb{E}\operatorname{diam}\left((x_\# + \ker(A)) \cap T\right)$$

$$\leq \frac{C\sigma^2 \omega(T)}{\sqrt{m}} \quad \blacksquare$$

Remark: So can achieve

$$\mathbb{E}\|\hat{x} - x_\#\|_2 \leq \varepsilon \operatorname{diam}(T)$$

using $m \geq C \frac{\sigma^4}{\varepsilon^2} d(T)$ measurements.

Often prior information encodes sparsity.

Define $\|x\|_0 = |\{i : x_i \neq 0\}|$

Suppose it's the case that

$$\|x_\#\|_0 =: s << n$$

This corresponds to setting

$$T = \{x \in \mathbb{R}^n : \|x\|_0 \leq s\}$$

Finding a point x satisfying

$$Ax = y \text{ and } x \in T \quad ①$$

can be difficult.

① It can be done efficiently under RIP assumptions by alternating projections

Instead, let's use a convex surrogate
Replace $\|x\|_0$ by $\|x\|_1$.

Intuition

$\lim_{p \downarrow 0} \|x\|_p = \|x\|_0$ and $\|x\|_1$ is the first true norm.

How big should the ℓ_1-ball be?

<u>Lemma:</u> $\{x \in \mathbb{R}^n : \|x\|_0 \le s, \ \|x\|_2 \le 1\} \subseteq \sqrt{s}\, \mathbb{B}_1^n$

<u>pf:</u> $\sum_{i=1}^n |x_i| \le \sum_{i \in \operatorname{supp} x} |x_i| \le \sqrt{s}\, \|x\|_2 \le \sqrt{s}$ $\boxtimes$

So new plan

(!) Find x: $y = Ax$ and $x \in \sqrt{s}\, \mathbb{B}_1^n$

<u>Thm:</u> Assume $\|x_\#\|_0 \le s$ and $\|x_\#\|_2 \le 1$
Then any solution of (!) satisfies

$$\mathbb{E}\|\hat{x} - x_\#\|_2 \le C\sigma^2 \sqrt{\frac{s \log(n)}{m}}$$

<u>pf:</u> Set $T = \sqrt{s}\, \mathbb{B}_1^n$. Then $x_\#, \hat{x} \in T$ and therefore

$$\mathbb{E}\|\hat{x} - x_\#\|_2 \le \frac{C\sigma^2 \omega(T)}{\sqrt{m}} = C\sigma^2 \sqrt{\frac{s \log n}{m}}$$ $\boxtimes$

So recovery is possible if

$$m \sim s \log(n)$$

Remark: dependence can be improved to

$$m \sim s \log(n/s)$$

Notice that (!) requires to know s and the theorem requires $\|x_\#\|_2 \leq 1$. We can do better.

Thm: Suppose $\|x_\#\|_0 \leq s$. Then any solution $\hat{x}$ of

$$\boxed{\begin{array}{ll} \min_x & \|x\|_1 \\ \text{s.t.} & Ax = y \end{array}}$$

satisfies

$$\mathbb{E}\|\hat{x} - x_\#\|_2 \leq c\sqrt{\frac{s \log n}{m}}\,\|x_\#\|_2$$

pf: We know $A\hat{x} = Ax = y$

Also $\|x_\#\|_1 \leq \sqrt{s}\cdot\|x\|_2$

$\Rightarrow \|\hat{x}\|_1 \leq \sqrt{s}\,\|x\|_2.$

$\hat{x}, x_\#$ satisfy $Ax' = y$, $x' \in \sqrt{s}\,\|x\|_2\, B_1^n$

$$\Longrightarrow \quad \mathbb{E}\|\hat{x} - x_\#\|_2 \leq C\sigma^2\sqrt{\frac{2\log n}{m}}\cdot\|x\|_2 \ \blacksquare$$

The previous results extend easily to <u>low-rank matrix recovery</u>.

<u>Model</u>: There is $X_\# \in \mathbb{R}^{d\times d}$ satisfying

$$y_i = \langle A_i, X_\# \rangle \quad \text{for } i=1,\dots,m$$

where $A_i \in \mathbb{R}^{d\times d}$ are independent.

The <u>side information</u> we assume is $\operatorname{rank}(X_\#) =: r \ll d$

Let $s(X)$ be the singular values of X.

Then $\operatorname{rank}(X) = \|s(X)\|_0$

$$\sum_{i=1}^{d} s_i(X) = \|X\|_* = \|s(X)\|_1$$

<u>Lemma</u>: $\{X : \operatorname{rank}(X) \le r, \|X\|_F \le 1\} \subset \sqrt{r}\, \mathbb{B}_*$

pf: $\|X\|_* = \sum_{i=1}^{r} s_i(X) \le \sqrt{r}\, \|s(X)\|_2 = \sqrt{r}\, \|X\|_F \le \sqrt{r}$ $\square$

Lemma: $\omega(\mathbb{B}_*) \le c\sqrt{d}$

pf: $$\mathbb{E} \sup_{\|X\|_* \le 1} \langle G, X \rangle = \mathbb{E} \|G\|_2 \le c\sqrt{d}$$ $\boxtimes$

Thm: Suppose that A_i are independent and $\mathrm{vec}(A_i)$ are σ-subGaussian, isotropic. Assume $X_\#$ satisfies $\|X_\#\|_F \le 1$. Then any $\hat{X}$ satisfying

$$y_i = \langle A_i, \hat{X} \rangle \quad \forall i = 1, \dots, m, \quad \|\hat{X}\|_* \le \sqrt{r}$$

satisfies

$$\mathbb{E}\|\hat{X} - X_\#\|_F \le c\sigma^2 \sqrt{\frac{rd}{m}}$$

pf: Set $T = \sqrt{r}\,\mathbb{B}_*$. Then we can treat $X, \hat{X}$ as long vectors and then

$$\mathbb{E}\|\hat{X} - X_\#\|_F \le \frac{C\sigma^2 \omega(T)}{\sqrt{m}} = C\sigma^2\sqrt{\frac{rd}{m}}$$ $\boxtimes$

As in the vector case, we can instead solve

$$\boxed{\begin{array}{ll} \min\limits_{X} & \|X\|_* \\ \text{s.t.} & y_i = \langle A_i, X \rangle \quad \forall i=1,\dots,m. \end{array}}$$

<u>Exact Recovery Guarantees</u>:

Consider the sparse recovery problem

$$\begin{array}{ll} \min & \|x\|_1 \\ \text{s.t.} & Ax=b \end{array}$$

We will now show that with h.p. $\hat{x} = x_\#$.

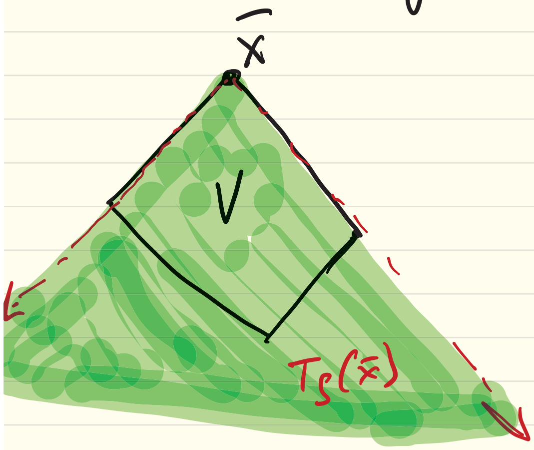
<u>Convex Optimization</u>:

Consider $\begin{array}{ll} \min & f(x) \\ \text{s.t.} & Ax=b \end{array}$

where f is convex. Then $\bar{x}$ is optimal <u>iff</u> $\left\{ \begin{array}{c} A\bar{x}=b \\ f'(\bar{x};v) \geq 0 \quad \forall v \in \ker A \end{array} \right\}$

For any convex set $V \subseteq \mathbb{R}^n$ and $\bar{x} \in V$ define the tangent cone $T_V(\bar{x}) = \mathrm{cl}\, \mathbb{R}_+(V - \bar{x})$

Lemma:
As long as $\bar{x}$ is not a minimizer of f,
have
$$T_V(\bar{x}) = \{v : f'(\bar{x}, v) \leq 0\}$$
where $V = \{y : f(y) \leq f(\bar{x})\}$



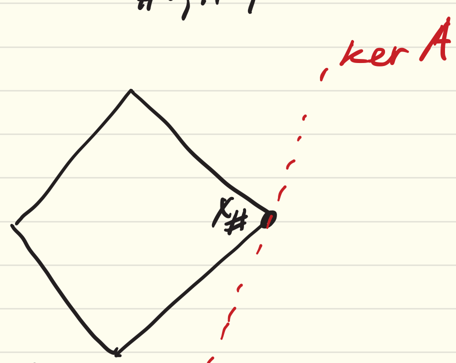
So conditions
$$\begin{cases} A\bar{x} = b \\ T_{[f(x) \leq f(\bar{x})]}(\bar{x}) \cap \ker(A) = \{0\} \end{cases}$$
guarantees $\bar{x}$ is a soln, and is in fact the <u>unique soln</u>.

So to show that $x_\#$ is the unique solution of

$$\min \|x\|_1 \quad \text{s.t.} \quad Ax=b$$

we estimate probability

$$T_{\|x_\#\|_1 \mathbb{B}_1}(x_\#) \cap \ker A = \{0\}$$



Define $Q := T_{\|x_\#\|_1 \mathbb{B}_1}(x_\#)$

Recall by the escape thm, the probability is governed by

$$w(Q \cap S^{n-1})$$

Lemma: $\|h\|_1 \le 2\sqrt{s}\,\|h\|_2 \quad \forall h \in Q.$

pf: Let $S = \mathrm{supp}(x_\#)$. Let's verify

$$\|h_{S^c}\|_1 \le \|h_S\|_1 \quad \forall h \in Q$$

Let $x \in \|x_\#\|_1 B_1$ and define $h = x - x_\#$.

Then

$$\|x_\#\|_1 \ge \|x\|_1 = \|x_\# + h\|_1 = \|(x_\#)_S + h_S\|_1 + \|(x_\#)_{S^c} + h_{S^c}\|_1$$

$$\ge \|(x_\#)_S\|_1 - \|h_S\|_1 + \|h_{S^c}\|_1$$

$\Rightarrow \|h_S\|_1 \ge \|h_{S^c}\|_1$ ok!

So $\|h\|_1 = \|h_S\|_1 + \|h_{S^c}\|_1 \le 2\|h_S\|_1 \le 2\sqrt{s}\,\|h\|_2$ ∎

So we learned

$$\boxed{Q \cap B_2 \subseteq 2\sqrt{s}\, B_1}$$

Thm: Suppose rows of $A \in \mathbb{R}^{m \times n}$ are independent, isotropic, σ-subGaussian. Then as long as

$$m \geq C\sigma^4 s \log n$$

then with probability $1-2\exp\left(-\frac{cm}{\sigma^4}\right)$, $x_\#$ is the unique soln of

$$\min_x \|x\|_1 \quad \text{s.t. } Ax = b$$

pf: The escape thm shows that as long as

$$m \geq C\sigma^4 \omega^2(Q \cap S^{n-1})$$

then

$$(Q \cap S^{n-1}) \cap \ker A = \emptyset$$

w.p. $1-2\exp(-cm/\sigma^4)$.

From the lemma

$$\omega(Q \cap \mathbb{B}_2) \leq c \cdot \omega(\sqrt{s}\,\mathbb{B}_1) \leq C\sqrt{s\log(n)}$$

$\boxtimes$

We next look at deterministic conditions for exact recovery.

Looking back at the previous proof, define

$$\mathbb{C}(S) := \{ h \in \mathbb{R}^n : \|h_{S^c}\|_1 \leq \|h_S\|_1 \}$$

Recall we showed:

$$T_{\|x_\#\|_1 \mathbb{B}_1}(x_\#) \subseteq \mathbb{C}(S)$$

where $S = \mathrm{supp}(x_\#)$.

Defn: $A \in \mathbb{R}^{m \times n}$ satisfies the <u>restricted nullspace property</u> with respect to $S \subseteq \{1, \dots, n\}$ if $\mathbb{C}(S) \cap \ker(A) = \{0\}$.

Thm: The following properties are equivalent.

ⓐ $A \in \mathbb{R}^{m \times n}$ satisfies the restricted nullspace property with respect to S

ⓑ for any $x_\# \in \mathbb{R}^n$ with $\text{supp}(x_\#) = S$, the problem

$$\min \|x\|_1 \quad \text{s.t.} \quad Ax = b$$

has the unique soln $x_\#$

pf: We already proved ⓐ $\Rightarrow$ ⓑ

To see converse, fix $\bar{x} \in \ker A \setminus \{0\}$.

Then $$\min \|x\|_1 \quad \text{s.t.} \quad Ax = A\begin{bmatrix} \bar{x}_S \\ 0 \end{bmatrix}$$

has $\begin{bmatrix} \bar{x}_S \\ 0 \end{bmatrix}$ as the unique optimal soln.

Since $A\bar{x} = 0$, it follows $\begin{bmatrix} 0 \\ -\bar{x}_{S^c} \end{bmatrix}$ is also feasible.

$\Rightarrow \|\bar{x}_S\|_1 < \|\bar{x}_{S^c}\|_1 \Rightarrow \bar{x} \notin \mathcal{C}(S)$ $\square$

Recall[‡] we also showed

$$\|h\|_1 \le 2\sqrt{s}\,\|h\|_2 \quad \forall_{h \in \mathbb{C}(S)}$$

So $\quad \mathbb{C}(S) \cap \mathbb{B}_2 \subset 2\sqrt{|S|}\,\mathbb{B}_1$

and escape thm show that if $A \in \mathbb{R}^{m \times n}$ is random (independent, subGaussian, isotropic rows) and

$$m \gtrsim C\sigma^4 \sqrt{|S|}\log(n)$$

then w.p. $1 - 2\exp(-\frac{cm}{\sigma^4})$ have

$$\mathbb{C}(S) \cap \ker A = \{0\}$$

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‡ [Recall $\forall_{h \in \mathbb{C}(S)}$

$\|h\|_1 = \|h_S\|_1 + \|h_{S^c}\|_1 \le 2\|h\|_S \le 2\sqrt{|S|}\,\|h\|_2$]

Q: How to verify the restricted nullspace property?

Here's a simpler condition

Defn: $A \in \mathbb{R}^{m \times n}$ satisfies the restricted isometry property relative to $S \in \{1, \dots, n\}$ with constant $\alpha, \beta > 0$, if

$$\alpha \|x\|_2 \leq \frac{1}{\sqrt{m}} \|Ax\|_2 \leq \beta \|x\|_2$$

for all x with $\|x\|_0 \leq S$.

Exercise: This is equivalent to requiring $\{\text{singular values of } \frac{1}{\sqrt{m}} A_{S'}\} \subset [\alpha, \beta]$ where $A_{S'}$ is submatrix of colums S'

Thm: If $\frac{2}{3} < \alpha < \beta < \frac{4}{3}$, then A satisfies the restricted nullspace property for any $S \subseteq \{1, \dots, n\}$ with $|S| \leq s$

[See Wainwright Proposition 7.11]

Thm: Suppose $A \in \mathbb{R}^{m \times n}$ satisfies RIP with parameters $\alpha, \beta, (1+\lambda)s$ where $\lambda > \left(\frac{\beta}{\alpha}\right)^2$. Then every s-sparse $x \in \mathbb{R}^n$ can be recovered exactly by solving

$$\begin{aligned} \min \quad & \|x\|_1 \\ \text{s.t.} \quad & Ax = b \end{aligned} \qquad (\ast)$$

pf: Define $h = \hat{x} - x$ where $\hat{x}$ solves $(\ast)$. Let $S_0 = \mathrm{supp}(x)$, let S_1 index λs largest elements of $h_{S_0^c}$ in absolute value, let S_2 index the next λs largest in $h_{S_0^c}$, and so on.

Set $S_{0,1} = S_0 \cup S_1$.

Since $Ah = A\hat{x} - Ax_0 = 0$, we have

$$0 = \|Ah\|_2 \geq \|A_{S_{0,1}} h_{S_{0,1}}\|_2 - \|A_{S_{0,1}^c} h_{S_{0,1}^c}\|_2$$

Since $|S_{0,1}| \leq s + \lambda s$, RIP gives

$$\|A_{S_{0,1}} h_{S_{0,1}}\|_2 \geq \alpha\sqrt{m} \cdot \|h_{S_{0,1}}\|_2$$

and

$$\|A_{S_{0,1}^c} h_{S_{0,1}^c}\|_2 \leq \sum_{i \geq 2} \|A_{S_i} h_{S_i}\|_2$$

$$\leq \beta\sqrt{m} \sum_{i \geq 2} \|h_{S_i}\|_2$$

So

$$\boxed{\alpha \|h_{0,1}\|_2 \leq \beta \sum_{i \geq 2} \|h_{S_i}\|_2}$$

Notice

$$\left\{\text{each entry of } h_{S_j}\right\} \le \frac{\|h_{S_{j-1}}\|_1}{\lambda s}$$

for $j \ge 2$. [For simplicity assume n divides λs]

So $$\|h_{S_j}\|_2 \le \frac{1}{\sqrt{\lambda s}} \|h_{S_{j-1}}\|_1$$

$$\Rightarrow \sum_{i \ge 2} \|h_{S_i}\|_2 \le \frac{1}{\sqrt{\lambda s}} \sum_{i \ge 1} \|h_i\|_1$$

$$= \frac{1}{\sqrt{\lambda s}} \|h_{S_0^c}\|_1$$

$$\le \frac{1}{\sqrt{\lambda s}} \|h_{S_0}\|_1$$ [$\mathcal{C}(S)$ contains tangent cone]

$$\le \frac{1}{\sqrt{\lambda}} \|h_{S_0}\|_2$$

$$\le \frac{1}{\sqrt{\lambda}} \|h_{S_{0,1}}\|_2$$

$$\Rightarrow \frac{\beta}{\sqrt{\lambda}} \|h_{S_{0,1}}\|_2 \ge \alpha \|h_{S_{0,1}}\|_2 \Rightarrow h = 0$$

$\boxtimes$

Thm: Let $A \in \mathbb{R}^{m \times n}$ have independent, isotropic, σ-subGaussian rows. Assume

$$m \geq \frac{\sigma^4 s}{\delta^2} \log(en/s)$$

Then with probability $1 - 2\exp\left(-\frac{c\delta^2 m}{\sigma^4}\right)$ have

$$(1-\delta)\|x\|_2 \leq \frac{1}{\sqrt{m}}\|Ax\|_2 \leq (1+\delta)\|x\|_2 \quad \forall x \text{ with } \|x\|_0 \leq s.$$

pf: We will control singular values of $A_{S'}$ uniformly over all S' with $|S'| = s$.

For any fixed S', we know

$$\sqrt{m} - r \leq \sigma_n(A_{S'}) \leq \sigma_1(A_{S'}) \leq \sqrt{m} + r$$

with probability $1 - 2\exp(-t^2)$ where $r = c\sigma^2(\sqrt{s} + t)$. Ensure

$\frac{r}{\sqrt{m}} \leq \delta$ by setting $t = \frac{\delta\sqrt{m}}{c\sigma^2} - \sqrt{s}$

Then $1 - \delta \leq \sigma_n(A_{S'}) \leq \sigma_1(A_{S'}) \leq 1 + \delta \quad \forall S' : |S'| = s$

w.p.

$$1 - 2\binom{n}{s}\exp(-t^2) \leq 1 - 2\exp\left(s\log\left(\frac{en}{s}\right) - t^2\right)$$

$$\leq 1 - 2\exp\left(-\frac{c\delta^2 m}{\sigma^4}\right) \quad \boxtimes$$

Sparse Recovery in the noisy setting

Suppose now that $x_\#$ satisfies

$$y = Ax_\# + w \quad \text{and} \quad \|x_\#\|_0 \leq s$$

and the noise vector $w \in \mathbb{R}^m$ is nonzero.

Three Problem Formulations:

(a) LASSO ("Least Absolute Shrinkage and Selection Operator")

$$\hat{x}_a \in \arg\min_x \left\{ \frac{1}{2m}\|y - Ax\|_2^2 + \lambda_m \|x\|_1 \right\}$$

(b) Basis Pursuit DeNoise (BPDN I)

$$\hat{x}_b \in \arg\min_x \frac{1}{2m}\|y - Ax\|_2^2 \text{ subject to } \|x\|_1 \leq R$$

(c) Basis Pursuit DeNoise (BPDN II)

$$\hat{x}_c \in \arg\min_x \|x\|_1 \text{ subject to } \frac{1}{2m}\|y - Ax\|_2^2 \leq b^2$$

Remark: Solutions are the same for appropriate choices of (λ_m, R, b) but one can not easily set R from b etc.

We will need to assume a modifies restricted nullspace property. Define

$$\mathbb{C}_\alpha(S) := \{ h \in \mathbb{R}^d : \|h_{S^c}\|_1 \leq \alpha \|h_S\|_1 \}$$

for some $\alpha \geq 0$.

<u>Defn:</u> $A \in \mathbb{R}^{m \times n}$ satisfies the <u>restricted eigenvalue (RE) condition</u> with parameters (κ, α) if

$$\frac{1}{m}\|Ah\|_2^2 \geq \kappa \|h\|_2^2 \quad \forall h \in \mathbb{C}_\alpha(S)$$

<u>Remark:</u>

$(\kappa, 1)$ is the restricted nullspace property.

<u>Intuition:</u> $\nabla^2\left(\frac{1}{2m}\|b - Ax\|_2^2\right) = \frac{1}{m} A^T A$ singular

RE ensures ∇^2 is positive definite in important directions.

<u>Thm</u>: Suppose $A \in \mathbb{R}^{m \times n}$ has independent, isotropic, σ-subGaussian rows. Then

$$\frac{1}{\sqrt{m}} \|Ah\|_2 \geq \left(1 - \delta - C\sigma^2 (1+\alpha) \sqrt{\frac{s \log n}{m}}\right) \|h\|_2 \quad \forall h \in \mathbb{C}_\alpha(S)$$

w.p. $1 - 2\exp\left(-\frac{c\delta^2 m}{\sigma^4}\right)$

<u>pf</u>: Matrix Deviation with $T = \mathbb{C}_\alpha(S) \cap S^{n-1}$ is

$$\sup_{x \in T} \left| \|Ax\|_2 - \sqrt{m} \|x\|_2 \right| \leq C\sigma^2 (w(T) + u)$$

w.p. $1 - 2\exp(-u^2)$.

Observe for $h \in T$ have

$$\|h\|_1 = \|h_S\|_1 + \|h_{S^c}\|_1 \leq (1+\alpha)\|h_S\|_1 \leq (1+\alpha)\sqrt{s}$$

$$\Rightarrow \omega(T) \leq c(1+\alpha)\sqrt{s \log(n)}$$

So $$\frac{1}{\sqrt{m}} \|Ax\|_2 \geq 1 - c\sigma^2 \left((1+\alpha) \sqrt{\frac{s \log n}{m}} + \frac{u}{\sqrt{m}} \right)$$

w.p. $1 - 2\exp(-u^2)$. Set $u = \frac{\delta \sqrt{m}}{c\sigma^2}$ $\boxtimes$

Thm: Suppose $\mathrm{supp}(x_\#) = S$ with $\|x_\#\|_0 \le s$, and A satisfies the RE condition $(\kappa, 3)$

ⓐ If $\lambda_m \ge 2 \left\| \frac{A^T w}{m} \right\|_\infty$, then

$$\|\hat{x}_a - x_\#\|_2 \le \frac{3}{\kappa} \sqrt{s}\, \lambda_m$$

ⓑ If $R = \|x_\#\|_1$, then

$$\|\hat{x}_b - x_\#\|_2 \le \frac{4}{\kappa} \sqrt{s} \left\| \frac{A^T w}{m} \right\|_\infty$$

ⓒ If $b^2 \ge \frac{\|w\|_2^2}{2m}$, then

$$\|\hat{x}_c - x_\#\|_2 \le \frac{4}{\kappa} \sqrt{s} \left\| \frac{A^T w}{m} \right\|_\infty + \frac{2}{\sqrt{\kappa}} \sqrt{b^2 - \frac{\|w\|_2^2}{2m}}$$

In all cases $\|\hat{x} - x_\#\|_1 \le 4\sqrt{s}\, \|\hat{x} - x_\#\|_2$

Example: Suppose $w \sim N(0, \sigma^2)$ and X fixed.

ⓐ Set $\lambda_m \asymp \sigma\left(\sqrt{\frac{\log n}{m}} + \delta\right)$

$$\Rightarrow \|\hat{x}_1 - x_\#\|_2 \lesssim \frac{\sigma}{\kappa} \cdot \sqrt{s} \left(\sqrt{\frac{\log n}{m}} + \delta\right)$$

w.p. $1 - 2\exp\left(\frac{m\delta^2}{2}\right)$

ⓑ, ⓒ are similar. λ and b are easy to estimate.

Example: Suppose $A_{ij} \sim N(0,1)$ and $\|w\|_\infty \le \sigma$

Same guarantees for LASSO:

pf: (b) Let $R = \|x_\#\|_1$ and let $\hat{x}$ be a minimizer. Then

$$\frac{1}{2m}\|y - A\hat{x}\|_2^2 \le \frac{1}{2m}\|y - Ax_\#\|_2^2$$

Plugging in $y = Ax_\# + w$

$$\Rightarrow \frac{\|Ah\|_2^2}{m} \le \frac{2w^TAh}{m} \quad \text{where } h = x - x_\#.$$

Since $h \in T_{RB_1}(x_\#)$ we know $h \in C_1(S)$
$\subseteq C_\alpha(S)$

$$\Rightarrow \chi \|h\|_2^2 \le 2\|h\|_1 \cdot \left\|\frac{A^Tw}{m}\right\|_\infty \le 2\sqrt{s}\,\|h\|_2 \left\|\frac{A^Tw}{m}\right\|_\infty \quad \square$$

(c) Observe $\frac{1}{2m}\|y - Ax_\#\|_2^2 = \frac{\|w\|^2}{2m} \le b^2$

So $x_\#$ is feasible $\Rightarrow \|\hat{x}\|_1 \le \|x_\#\|_1$

$\Rightarrow h = \hat{x} - x_\# \in C_1(S)$.

Observe $\frac{1}{2m}\|y - A\hat{x}\|_2^2 \le b^2 = \frac{1}{2m}\|y - Ax_\#\|_2^2 + \left(b^2 - \frac{\|w\|_2^2}{m}\right)$

Rearranging

$$\frac{\|Ah\|_2^2}{m} \leq 2\,\frac{w^T A h}{m} + 2\left(b^2 - \frac{\|w\|^2}{2m}\right)$$

$$\Rightarrow \kappa \|h\|_2^2 \leq 4\sqrt{s}\,\|h\|_2 \left\|\frac{A^T w}{m}\right\|_\infty + 2\left(b^2 - \frac{\|w\|^2}{2m}\right)$$

$$\leq \max\left\{4\sqrt{s}\,\|h\|_2 \left\|\frac{A^T w}{m}\right\|_\infty,\ 2\left(b^2 - \frac{\|w\|^2}{2m}\right)\right\}$$

© Define

$$L(x, \lambda_m) = \frac{1}{2n}\|y - Ax\|_2^2 + \lambda_m \|x\|_1$$

We know

$$L(\hat{x}, \lambda_m) \leq L(x_\#, \lambda_m) = \frac{1}{2}\|w\|_2^2 + \lambda_m \|x_\#\|_1$$

$$\Rightarrow \frac{1}{2m}\|Ah\|_2^2 \leq \frac{w^T A h}{m} + \lambda_m\left(\|x_\#\|_1 - \|\hat{x}\|_1\right)$$

We can write

$$\|x_\#\|_1 - \|\hat{x}\|_1 = \|(x_\#)_S\|_1 - \|x_{\#_S} + h_S\|_1 - \|h_{S^c}\|_1$$

$$\leq \|h_S\|_1 - \|h_{S^c}\|_1$$

So

$$0 \leq \frac{1}{m}\|Ah\|_2^2 \leq 2\|h\|_1 \left\|\frac{A^T w}{m}\right\|_\infty + 2\lambda_m\left(\|h_S\|_1 - \|h_{S^c}\|_1\right)$$

choice of λ_m $\leq \lambda_m\left(\|h\|_1 + 2\|h_S\|_1 - 2\|h_{S^c}\|_1\right) \leq \lambda_m\left(3\|h_S\|_1 - \|h_{S^c}\|_1\right)$

So $h \in \mathbb{C}_3(S)$. RE condition gives

$$\kappa \|h\|_2^2 \leq 3 \lambda_m \sqrt{s}\, \|h\|_2 \quad \square$$