# Convex Analysis and Nonsmooth Optimization

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# Contents

# Chapter 1

# Background

This chapter sets the notation and reviews the background material that will be used throughout the rest of the book. The reader can safely skim this chapter during the first pass and refer back to it when necessary. The discussion is purposefully kept brief. The comments section at the end of the chapter lists references where a more detailed treatment may be found.

**Roadmap.** Sections 1.1-1.3 review basic constructs of linear algebra, including inner products, norms, linear maps and their adjoints, as well as eigenvalue and singular value decompositions. Section 1.4 establishes notation for basic set operations, such as sums and images/preimages of sets. Section 1.5 focuses on topological preliminaries; the main results are the Bolzano-Weierstrass theorem and a variant of the extreme value theorem. The final Sections 1.6-1.7 formally define first and second-order derivatives of multivariate functions, establish estimates on the error in Taylor approximations, and deduce derivative-based conditions for local optimality. The material in Sections 1.6-1.7 is often covered superficially in undergraduate courses, and therefore we provide an entirely self-contained treatment.

## 1.1 Inner products and linear maps

Throughout, we fix an *Euclidean space* $\mathbf{E}$, meaning that $\mathbf{E}$ is a finite-dimensional real vector space endowed with an *inner product* $\langle \cdot, \cdot \rangle$. Recall that an inner-product on $\mathbf{E}$ is an assignment $\langle \cdot, \cdot \rangle \colon \mathbf{E} \times \mathbf{E} \to \mathbf{R}$ satisfying the following three properties for all $x, y, z \in \mathbf{E}$ and scalars $a, b \in \mathbf{R}$:

**(Symmetry)** $\langle x, y \rangle = \langle y, x \rangle$

**(Bilinearity)** $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

**(Positive definiteness)** $\langle x, x\rangle \geq 0$ and equality $\langle x, x\rangle = 0$ holds if and only if $x = 0$.

The most familiar example is the Euclidean space of $n$-dimensional column vectors $\mathbf{R}^n$, which we always equip with the *dot-product*

$$\langle x, y\rangle := \sum_{i=1}^{n} x_i y_i.$$

One can equivalently write $\langle x, y\rangle = x^T y$. A basic result of linear algebra shows that all Euclidean spaces $\mathbf{E}$ can be identified with $\mathbf{R}^n$ for some integer $n$, once an orthonormal basis is chosen. Though such a basis-specific interpretation can be useful, it is often distracting, with the indices hiding the underlying geometry. Consequently, it is often best to think coordinate-free.

The space of real $m \times n$-matrices $\mathbf{R}^{m\times n}$ furnishes another example of an Euclidean space, which we always equip with the *trace product*

$$\langle X, Y\rangle := \operatorname{tr} X^T Y.$$

Some arithmetic shows the equality $\langle X, Y\rangle = \sum_{i,j} X_{ij} Y_{ij}$. Thus the trace product on $\mathbf{R}^{m\times n}$ coincides with the usual dot-product on the matrices stretched out into long vectors. An important Euclidean subspace of $\mathbf{R}^{n\times n}$ is the space of real symmetric $n\times n$-matrices $\mathbf{S}^n$, along with the trace product $\langle X, Y\rangle := \operatorname{tr} XY$.

For any linear mapping $\mathcal{A}\colon \mathbf{E} \to \mathbf{Y}$, there exists a unique linear mapping $\mathcal{A}^*\colon \mathbf{Y} \to \mathbf{E}$, called the *adjoint*, satisfying

$$\langle \mathcal{A}x, y\rangle = \langle x, \mathcal{A}^* y\rangle \qquad \text{for all points} \qquad x \in \mathbf{E},\ y \in \mathbf{Y}.$$

In the most familiar case of $\mathbf{E} = \mathbf{R}^n$ and $\mathbf{Y} = \mathbf{R}^m$, any linear map $\mathcal{A}$ can be identified with a matrix $A \in \mathbf{R}^{m\times n}$, while the adjoint $\mathcal{A}^*$ may then be identified with the transpose $A^T$.

**Exercise 1.1.** Given a collection of real $m \times n$ matrices $A_1, A_2, \ldots, A_l$, define the linear mapping $\mathcal{A}\colon \mathbf{R}^{m\times n} \to \mathbf{R}^l$ by setting

$$\mathcal{A}(X) := (\langle A_1, X\rangle, \langle A_2, X\rangle, \ldots, \langle A_l, X\rangle).$$

Show that the adjoint is the mapping $\mathcal{A}^* y = y_1 A_1 + y_2 A_2 + \ldots + y_l A_l$.

Linear mappings $\mathcal{A}\colon \mathbf{E} \to \mathbf{E}$, between a Euclidean space $\mathbf{E}$ and itself, are called *linear operators*, and are said to be *self-adjoint* if equality $\mathcal{A} = \mathcal{A}^*$ holds. Self-adjoint operators on $\mathbf{R}^n$ are precisely those operators that are representable as symmetric matrices. A self-adjoint operator $\mathcal{A}$ is *positive semi-definite*, denoted $\mathcal{A} \succeq 0$, whenever

$$\langle \mathcal{A}x, x\rangle \geq 0 \quad \text{ for all } x \in \mathbf{E}.$$

Similarly, a self-adjoint operator $\mathcal{A}$ is *positive definite*, denoted $\mathcal{A} \succ 0$, whenever

$$\langle \mathcal{A}x, x\rangle > 0 \quad \text{ for all } 0 \neq x \in \mathbf{E}.$$

A positive semidefinite linear operator $\mathcal{A}$ is positive definite if and only if $\mathcal{A}$ is invertible.

## 1.2 Norms

A *norm* on a vector space $\mathcal{V}$ is a function $\|\cdot\|\colon \mathcal{V} \to \mathbf{R}$ for which the following three properties hold for all point $x, y \in \mathcal{V}$ and scalars $a \in \mathbf{R}$:

**(Absolute homogeneity)** $\|ax\| = |a| \cdot \|x\|$

**(Triangle inequality)** $\|x + y\| \leq \|x\| + \|y\|$

**(Positivity)** Equality $\|x\| = 0$ holds if and only if $x = 0$.

The inner product in the Euclidean space $\mathbf{E}$ always induces a norm $\|x\| = \sqrt{\langle x, x\rangle}$. Unless specified otherwise, the symbol $\|x\|$ for $x \in \mathbf{E}$ will always denote this induced norm. For example, the dot product on $\mathbf{R}^n$ induces the usual 2-norm $\|x\|_2 := \sqrt{x_1^2 + \ldots + x_n^2}$, while the trace product on $\mathbf{R}^{m\times n}$ induces the *Frobenius norm* $\|X\|_F := \sqrt{\operatorname{tr}(X^TX)}$.

Other important examples of norms are the $l_p$-*norms* on $\mathbf{R}^n$:

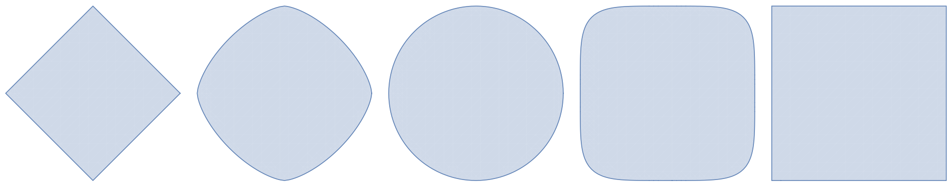
$$\|x\|_p = \begin{cases} (|x_1|^p + \ldots + |x_n|^p)^{1/p} & \text{for } 1 \leq p < \infty \\ \max\{|x_1|, \ldots, |x_n|\} & \text{for } p = \infty \end{cases}.$$

The most notable of these are the $l_1$, $l_2$, and $l_\infty$ norms; see Figure 1.1.

For an arbitrary norm $\|\cdot\|$ on $\mathbf{E}$, the dual norm $\|\cdot\|^*$ on $\mathbf{E}$ is defined by

$$\|v\|^* := \max\{\langle v, x\rangle : \|x\| \leq 1\}.$$

Thus $\|v\|^*$ is the maximal value that the linear function $x \mapsto \langle v, x\rangle$ takes over the closed unit ball of the norm $\|\cdot\|$. For example, the $l_p$ and $l_q$ norms



(a) $p = 1$ (b) $p = 1.5$ (c) $p = 2$ (d) $p = 5$ (e) $p = \infty$

Figure 1.1: Unit balls of $\ell_p$-norms.

on $\mathbf{R}^n$ are dual to each other whenever $p^{-1} + q^{-1} = 1$ and $p, q \in [1, \infty]$. In particular, the $\ell_2$-norm on $\mathbf{R}^n$ is self-dual; the same goes for the Frobenius norm on $\mathbf{R}^{m\times n}$ (why?). For an arbitrary norm $\|\cdot\|$ on $\mathbf{E}$, the Cauchy-Schwarz inequality holds:

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|^*.$$

**Exercise 1.2.** Given a positive definite linear operator $\mathcal{A}$ on $\mathbf{E}$, show that the assignment $\langle v, w\rangle_{\mathcal{A}} := \langle \mathcal{A}v, w\rangle$ is an inner product on $\mathbf{E}$, with the induced norm $\|v\|_{\mathcal{A}} = \sqrt{\langle \mathcal{A}v, v\rangle}$. Show that the dual norm with respect to the original inner product $\langle\cdot,\cdot\rangle$ is $\|v\|^*_{\mathcal{A}} = \|v\|_{\mathcal{A}^{-1}} = \sqrt{\langle \mathcal{A}^{-1}v, v\rangle}$.

All norms on $\mathbf{E}$ are "equivalent" in the sense that any two are within a constant factor of each other. More precisely, for any two norms $\rho_1(\cdot)$ and $\rho_2(\cdot)$, there exist constants $\alpha, \beta \geq 0$ satisfying

$$\alpha\rho_1(x) \leq \rho_2(x) \leq \beta\rho_1(x) \qquad \text{for all } x \in \mathbf{E}.$$

Case in point, for any vector $x \in \mathbf{R}^n$, the relations hold:

$$\begin{aligned} \|x\|_2 \leq \|x\|_1 &\leq \sqrt{n}\|x\|_2 \\ \|x\|_\infty \leq \|x\|_2 &\leq \sqrt{n}\|x\|_\infty \\ \|x\|_\infty \leq \|x\|_1 &\leq n\|x\|_\infty. \end{aligned}$$

For our purposes, the term "equivalent" is a misnomer: the proportionality constants $\alpha, \beta$ strongly depend on the (often enormous) dimension of the vector space $\mathbf{E}$. Hence measuring quantities in different norms can yield strikingly different conclusions.

Consider a linear map $\mathcal{A}\colon \mathbf{E} \to \mathbf{Y}$, and norms $\|\cdot\|_{\mathbf{E}}$ on $\mathbf{E}$ and $\|\cdot\|_{\mathbf{Y}}$ on $\mathbf{Y}$. We define the *induced norm* of $\mathcal{A}$ by

$$\|\mathcal{A}\|_{\mathbf{E},\mathbf{Y}} := \max_{x\colon \|x\|_{\mathbf{E}} \leq 1} \|\mathcal{A}x\|_{\mathbf{Y}}.$$

In particular, if $\|\cdot\|_{\mathbf{E}}$ and $\|\cdot\|_{\mathbf{Y}}$ are the norms induced by the inner products in $\mathbf{E}$ and $\mathbf{Y}$, then the corresponding matrix norm is called the *operator norm* of $\mathcal{A}$ and will be denoted simply by $\|\mathcal{A}\|_{\mathrm{op}}$.

**Exercise 1.3.** Equip $\mathbf{R}^n$ and $\mathbf{R}^m$ with the $l_p$-norms. Then for any matrix $A \in \mathbf{R}^{m\times n}$, show the equalities

$$\|A\|_{1,1} = \max_{j=1,\ldots,n} \|A_{\bullet j}\|_1 \qquad \text{and} \qquad \|A\|_{\infty,\infty} = \max_{i=1,\ldots,n} \|A_{i\bullet}\|_1,$$

where $A_{\bullet j}$ and $A_{i\bullet}$ denote the $j$'th column and $i$'th row of $A$, respectively.

## 1.3 Eigenvalue and singular value decompositions of matrices

The symbol $\mathbf{S}^n$ will denote the set of $n \times n$ real symmetric matrices

$$\mathbf{S}^n := \{X \in \mathbf{R}^{n\times n} : X^T = X\},$$

while $O(n)$ will denote the set of $n \times n$ real orthogonal matrices:

$$O(n) := \{X \in \mathbf{R}^{n\times n} : X^T X = XX^T = I\}.$$

A number $\lambda \in \mathbf{R}$ is an *eigenvalue* of a symmetric matrix $A \in \mathbf{S}^{n\times n}$ if there exists a vector $0 \neq v \in \mathbf{R}^n$ satisfying $Av = \lambda v$. Any such vector $v$ is called an *eigenvector* corresponding to $\lambda$. Thus the eigenvalues of $A$ are precisely the roots of the *characteristic polynomial*

$$\lambda \mapsto \det(A - \lambda I).$$

A central result of linear algebra shows that all $n$ roots of this polynomial are real, when $A$ is symmetric. We may therefore fix an ordering and denote the eigenvalues of $A$ by

$$\lambda_1(A) \geq \lambda_1(A) \geq \ldots \geq \lambda_n(A).$$

The Rayleigh-Ritz theorem shows that the following relation always holds:

$$\lambda_n(A) \leq \frac{\langle Au, u\rangle}{\langle u, u\rangle} \leq \lambda_1(A) \qquad \text{for all } u \in \mathbf{R}^n \setminus \{0\}.$$

Thus the two conditions, $A \succeq 0$ and $\lambda_n(A) \geq 0$ are equivalent; similarly, $A \succ 0$ if and only $\lambda_n(A) > 0$.

Any symmetric matrix $A \in \mathbf{S}^n$ admits an *eigenvalue decomposition*, meaning a factorization of the form

$$A = U\Lambda U^T,$$

where $U \in O(n)$ is orthogonal and $\Lambda \in \mathbf{S}^n$ is a diagonal matrix. The diagonal elements of $\Lambda$ are precisely the eigenvalues of $A$ and the columns of $U$ are corresponding eigenvectors.

More generally, any rectangular matrix $A \in \mathbf{R}^{m\times n}$ admits a *singular value decomposition*, meaning a factorization of the form

$$A = U\Sigma V^T,$$

where $U \in O(m)$ and $V \in O(n)$ are orthogonal matrices and $\Sigma \in \mathbf{R}^{m\times n}$ is a diagonal matrix with nonnegative diagonal entries. The diagonal elements of $\Sigma$ are uniquely defined and are called the *singular values* of $A$. Supposing without loss of generality $m \le n$, the singular values of $A$ are precisely the square roots of the eigenvalues of $AA^T$, and we denote them by

$$\sigma_1(A) \ge \sigma_2(A) \ge \ldots \ge \sigma_m(A) \ge 0.$$

In particular, the operator norm $\|A\|_{\mathrm{op}}$ of any matrix $A \in \mathbf{R}^{m\times n}$ equals its maximal singular-value $\sigma_1(A)$. See Figure 1.2 for an illustration.

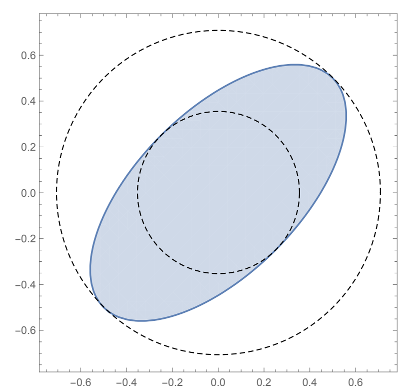



Figure 1.2: The shaded ellipse is the image of the unit disk by a nonsingular matrix $A \in \mathbf{R}^{2\times 2}$. The radii of the circumscribed and inscribed circles are $\sigma_1(A)$ and $\sigma_2(A)$, respectively.

## 1.4 Set operations

In this section, we review notation for sums, generated cones, and images/preimages of sets. For any two sets $A, B \subset \mathbf{E}$ and $\lambda \in \mathbf{R}$, define the set operations:

$$\lambda A := \{\lambda a : a \in A\} \qquad \text{and} \qquad A + B := \{a + b : a \in A,\ b \in B\}.$$

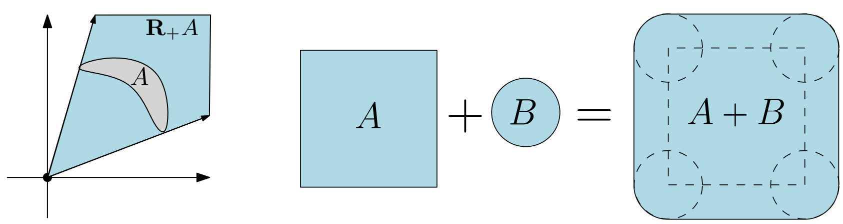
Thus the points in $\lambda A$ are simply the points in $A$ scaled by $\lambda$. One can visualize the sum $A + B$ by writing it more suggestively as

$$A + B = \bigcup_{a \in A} (a + B).$$

Thus $A+B$ is formed from the union of the shifted sets $a+B$ over all points $a \in A$. In particular, forming the sum of a set $A \subset \mathbf{E}$ and a unit ball $\mathbb{B} \subset \mathbf{E}$ has the affect of "fattening" $A$. The symbol $A - B$ is defined similarly. The cone generated by a set $A \subset \mathbf{E}$ will be denoted by

$$\mathbf{R}_+ A := \{\lambda x : x \in A, \lambda \geq 0\}.$$

See Figure 1.3 for an illustration of the generated cone and sum operation.




(a) Generated cone.

(b) Disk plus square.

Figure 1.3: Sum and cone operations.

For any map $\mathcal{F} \colon \mathbf{E} \to \mathbf{Y}$ and sets $A \subset \mathbf{E}$ and $B \subset \mathbf{Y}$, define the two sets

$$\mathcal{F}A = \{\mathcal{F}(x) : x \in A\} \qquad \text{and} \qquad \mathcal{F}^{-1}B = \{x : \mathcal{F}x \in B\}.$$

The set $\mathcal{F}A$ is called the image of $A$ under $\mathcal{F}$, while $\mathcal{F}^{-1}B$ is called the preimage of $B$ under $\mathcal{F}$. Notice that the sum $A + B$ can also be written as the linear image of the product set $Q := A \times B$ under the map $\mathcal{F}(x, y) = x + y$.

## 1.5 Point-set topology and existence of minimizers

The symbol $B_r(x)$ will denote an open ball of radius $r$ around a point $x$, namely $B_r(x) := \{y \in \mathbf{E} : \|y - x\| < r\}$. We will denote the open unit ball by $\mathbb{B}$. The *closure* of a set $Q \subset \mathbf{E}$, denoted $\operatorname{cl} Q$, consists of all points $x$ such that the ball $B_\epsilon(x)$ intersects $Q$ for all $\epsilon > 0$; the *interior* of $Q$, written as $\operatorname{int} Q$, is the set of all points $x$ such that $Q$ contains some open ball around $x$. We say that $Q$ is an *open set* if it coincides with its interior and a *closed set* if it coincides with its closure. Any set $Q$ in $\mathbf{E}$ that is closed and bounded is called a *compact set*. We will often use the following result without explicitly quoting it.

**Theorem 1.4** (Bolzano-Weierstrass)**.** *Any sequence in a compact set $Q \subset \mathbf{E}$ admits a subsequence converging to a point in $Q$.*

It will often be convenient to allow functions to take infinite values. Consequently, define the *extended real line* $\overline{\mathbf{R}} := \mathbf{R} \cup \{\pm\infty\}$. A basic question one can ask when minimization a function $f : \mathbf{E} \to \overline{\mathbf{R}}$ is whether a minimizer even exists. For example, the infimal value of the function $f(x) = e^x$ is zero and yet this value is not attained at any point. A standard way to ensure that a function has minimizers, which we now discuss, is by assuming (1) compactness and (2) a mild continuity property.

**Definition 1.5** (Lower-semicontinuous)**.** A function $f : \mathbf{E} \to \overline{\mathbf{R}}$ is *lower-semicontinuous at* $x \in \mathbf{E}$ if the inequality $\liminf_{y \to x} f(y) \geq f(x)$ holds. If $f$ is lower-semicontinuous at every point $x \in \mathbf{E}$, then we call $f$ *closed*.

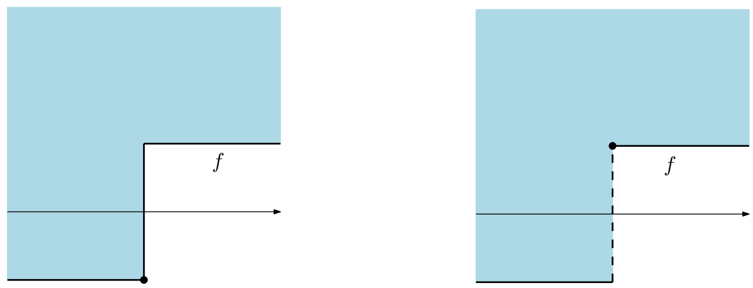
Intuitively, lower-semicontinuity of $f$ at $x$ asserts that the function values cannot suddenly jump down as one moves slightly away from $x$. For example, the step function

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

is not lower-semicontinuous at $x = 0$ since $\lim_{i \to \infty} f(i^{-1}) = -1 < f(0)$. If instead we redefine $f(0) = -1$, then the function becomes lower-semicontinuous; see Figure 1.4.

The following exercise shows that $f$ is lower-semicontinuous at every point in $\mathbf{E}$ if and only if its epigraph—the set above the graph—is a closed set, thereby explaining why Definition 1.5 calls such functions closed. The geometry of the epigraph will play a central role in the later chapters.

**Exercise 1.6.** ✍ Show that a function $f : \mathbf{E} \to \overline{\mathbf{R}}$ is closed if and only if the set, $\{(x, r) \in \mathbf{E} \times \mathbf{R} : f(x) \leq r\}$, is closed.




(a) Closed. (b) Not closed.

Figure 1.4: Closed functions.

The following exercise shows that the infimal value of a closed function on a compact set is always attained.

**Exercise 1.7** (Existence of minimizers on compact sets)**.** ✍ Consider a closed function $f\colon \mathbf{E} \to \overline{\mathbf{R}}$ and a compact set $Q \subset \mathbf{E}$. Then the infimum value $\inf_{x\in Q} f(x)$ is attained at some point in $Q$.

[**Hint:** Apply the Bolzano-Weierstrass Theorem to the sequence $x_i \in Q$ satisfying $f(x_i) \to \inf_Q f$ and invoke lower-semicontinuity.]

An important downside of the above exercise is it only guarantees existence of minimizers over compact sets. In light of the exponential example mentioned previously, if we wish to guarantee existence of minimizers over $\mathbf{E}$, then we must focus on a favorable class of functions.

**Definition 1.8** (Coercive)**.** A function $f\colon \mathbf{E} \to \overline{\mathbf{R}}$ is *coercive* if for any sequence $x_i$ with $\|x_i\| \to \infty$, it must be that $f(x_i) \to \infty$.

Equivalently, a function $f$ is coercive precisely when the sublevel sets $\{x : f(x) \le r\}$ are bounded for every $r \in \mathbf{R}$ (check this!). For example, the function $f(x) = e^{x^2}$ is coercive while the exponential $f(x) = e^x$ is not.

**Exercise 1.9** (Existence of unconstrained minimizers)**.** ✍ Any coercive closed function $f\colon \mathbf{E} \to \overline{\mathbf{R}}$ has a minimizer.
[**Hint:** Choose $r \in \mathbf{R}$ such that the sublevel set $\mathcal{L} = \{x : f(x) \le r\}$ is nonempty and apply Exercise 1.7.]

## 1.6 Differentiability

For the rest of the section, let $\mathbf{E}$ and $\mathbf{Y}$ be two Euclidean spaces, and $U$ an open subset of $\mathbf{E}$. A mapping $F\colon Q \to \mathbf{Y}$, defined on a subset $Q \subset \mathbf{E}$, is *continuous* at a point $x \in Q$ if for any sequence $x_i$ in $Q$ converging to $x$, the values $F(x_i)$ converge to $F(x)$. We say that $F$ is *continuous* if it is continuous at every $x \in Q$. We say that $F$ is *$L$-Lipschitz continuous* if

$$\|F(y) - F(x)\| \leq L\|y - x\| \quad \text{for all } x, y \in Q.$$

A function $f\colon U \to \mathbf{R}$ is *differentiable* at a point $x$ in $U$ if there exists a vector, denoted by $\nabla f(x) \in \mathbf{E}$, satisfying

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - \langle \nabla f(x), h\rangle}{\|h\|} = 0. \tag{1.1}$$

Rather than carrying such fractions around, which can be cumbersome, it is convenient to introduce the following notation. The symbol $o(r)$ will always stand for a term satisfying $0 = \lim_{r \downarrow 0} o(r)/r$. Then the equation (1.1) simply amounts to the expression

$$f(x+h) = f(x) + \langle \nabla f(x), h\rangle + o(\|h\|).$$

The vector $\nabla f(x)$ is called the *gradient* of $f$ at $x$. In the most familiar setting $\mathbf{E} = \mathbf{R}^n$, the gradient is simply the vector of partial derivatives

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix}.$$

If the gradient mapping $x \mapsto \nabla f(x)$ is well-defined and continuous on $U$, we say that $f$ is *$C^1$-smooth*. If the stronger property

$$\|\nabla f(y) - \nabla f(x)\|^* \leq \beta\|y - x\| \qquad \text{holds for all } x, y \in U,$$

then we say that $f$ is *$\beta$-smooth*. Recall that $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{E}$ and $\|\cdot\|^*$ is the dual norm.

More generally, a mapping $F\colon U \to \mathbf{Y}$ is *differentiable* at $x \in U$ if there exists a linear mapping from $\mathbf{E}$ to $\mathbf{Y}$, denoted by $\nabla F(x)$, satisfying

$$F(x+h) = F(x) + \nabla F(x)h + o(\|h\|).$$

The linear mapping $\nabla F(x)$ is called the *Jacobian* of $F$ at $x$. If the assignment $x \mapsto \nabla F(x)$ is continuous, we say that $F$ is *$C^1$-smooth*. In the most familiar setting $\mathbf{E} = \mathbf{R}^n$ and $\mathbf{Y} = \mathbf{R}^m$, we can write $F$ in terms of coordinate functions $F(x) = (F_1(x), \ldots, F_m(x))$, and then the Jacobian is simply

$$\nabla F(x) = \begin{pmatrix} \nabla F_1(x)^T \\ \nabla F_2(x)^T \\ \vdots \\ \nabla F_m(x)^T \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1(x)}{\partial x_1} & \frac{\partial F_1(x)}{\partial x_2} & \cdots & \frac{\partial F_1(x)}{\partial x_n} \\ \frac{\partial F_2(x)}{\partial x_1} & \frac{\partial F_2(x)}{\partial x_2} & \cdots & \frac{\partial F_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m(x)}{\partial x_1} & \frac{\partial F_m(x)}{\partial x_2} & \cdots & \frac{\partial F_m(x)}{\partial x_n} \end{pmatrix}.$$

Finally, we introduce second-order derivatives. A $C^1$-smooth function $f \colon U \to \mathbf{R}$ is *twice differentiable* at a point $x \in U$ if the gradient map $\nabla f \colon U \to \mathbf{E}$ is differentiable at $x$. Then the Jacobian of the gradient $\nabla(\nabla f)(x)$ is denoted by $\nabla^2 f(x)$ and is called the *Hessian* of $f$ at $x$. Unraveling notation, the Hessian $\nabla^2 f(x)$ is characterized by the condition

$$\nabla f(x + h) = \nabla f(x) + \nabla^2 f(x)h + o(\|h\|).$$

If the map $x \mapsto \nabla^2 f(x)$ is continuous, we say that $f$ is $C^2$-smooth. If $f$ is indeed $C^2$-smooth, then a basic result of calculus shows that $\nabla^2 f(x)$ is a self-adjoint operator.

In the standard setting $\mathbf{E} = \mathbf{R}^n$, the Hessian is the matrix of second-order partial derivatives

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_1(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{pmatrix}.$$

This matrix is symmetric, as long as it varies continuously with $x$ in $U$.

**Exercise 1.10.** ✍ Define the function

$$f(x) = \tfrac{1}{2}\langle \mathcal{A}x, x\rangle + \langle v, x\rangle + c$$

where $\mathcal{A} \colon \mathbf{E} \to \mathbf{E}$ is a linear operator, $v$ lies in $\mathbf{E}$, and $c$ is a real number.

1. Show that if $\mathcal{A}$ is replaced by the self-adjoint operator $(\mathcal{A} + \mathcal{A}^*)/2$, the function values $f(x)$ remain unchanged.
2. Assuming $\mathcal{A}$ is self-adjoint, derive the equations:

$$\nabla f(x) = \mathcal{A}x + v \quad \text{and} \quad \nabla^2 f(x) = \mathcal{A}.$$

3. Assuming $\mathcal{A}$ is self-adjoint, show that $f$ is coercive if and only if $\mathcal{A}$ is positive definite.

**Exercise 1.11.** Define the function $f(x) = \frac{1}{2}\|F(x)\|^2$, where $F\colon \mathbf{E} \to \mathbf{Y}$ is a $C^1$-smooth mapping. Prove the identity $\nabla f(x) = \nabla F(x)^* F(x)$.

**Exercise 1.12.** ✍ Consider a function $f\colon U \to \mathbf{R}$ and a linear mapping $\mathcal{A}\colon \mathbf{Y} \to \mathbf{E}$ and define the composition $h(x) = f(\mathcal{A}x)$.

1. Show that if $f$ is differentiable at $\mathcal{A}x$, then
$$\nabla h(x) = \mathcal{A}^* \nabla f(\mathcal{A}x).$$
2. Show that if $f$ is twice differentiable at $\mathcal{A}x$, then
$$\nabla^2 h(x) = \mathcal{A}^* \nabla^2 f(\mathcal{A}x)\mathcal{A}.$$

**Exercise 1.13.** ✍ Define the two sets
$$\begin{aligned} \mathbf{R}^n_{++} &:= \{x \in \mathbf{R}^n : x_i > 0 \text{ for all } i = 1, \ldots, n\}, \\ \mathbf{S}^n_{++} &:= \{X \in \mathbf{S}^n : X \succ 0\}. \end{aligned}$$
Consider the two functions $f\colon \mathbf{R}^n_{++} \to \mathbf{R}$ and $F\colon \mathbf{S}^n_{++} \to \mathbf{R}$ given by
$$f(x) = -\sum_{i=1}^n \log x_i \qquad \text{and} \qquad F(X) = -\ln\det(X),$$
respectively. Note, from basic properties of the determinant, the equality $F(X) = f(\lambda(X))$, where we set $\lambda(X) := (\lambda_1(X), \ldots, \lambda_n(X))$.

1. Find the derivatives $\nabla f(x)$ and $\nabla^2 f(x)$ for $x \in \mathbf{R}^n_{++}$.
2. Using the property $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$, prove $\nabla F(X) = -X^{-1}$ and $\nabla^2 F(X)[V] = X^{-1}VX^{-1}$ for any $X \succ 0$.

   [**Hint:** To compute $\nabla F(X)$, justify
   $$F(X+tV) - F(X) + t\langle X^{-1}, V\rangle = -\ln\det(I + X^{-1/2}VX^{-1/2}) + \mathrm{tr}\,(X^{-1/2}VX^{-1/2}).$$
   By rewriting the expression in terms of eigenvalues of $X^{-1/2}VX^{-1/2}$, deduce that the right-hand-side is $o(t)$. To compute the Hessian, observe
   $$(X+V)^{-1} = X^{-1/2}\left(I + X^{-1/2}VX^{-1/2}\right)^{-1} X^{-1/2},$$
   and then use the expansion
   $$(I+A)^{-1} = I - A + A^2 - A^3 + \ldots = I - A + O(\|A\|^2_{\mathrm{op}}),$$
   whenever $\|A\|_{\mathrm{op}} < 1$. ]

3. Show
$$\langle \nabla^2 F(X)[V], V\rangle = \|X^{-\frac{1}{2}} V X^{-\frac{1}{2}}\|_F^2$$
for any $X \succ 0$ and $V \in \mathbb{S}^n$. Deduce that the operator $\nabla^2 F(X) \colon \mathbf{S}^n \to \mathbf{S}^n$ is positive definite.

## 1.7 Accuracy in approximation and optimality conditions

A set $Q$ in $\mathbf{E}$ is *convex* if for any two points $x, y \in Q$ and real $\lambda \in [0,1]$, the point $\lambda x + (1-\lambda)y$ lies in $Q$. In other words, a set $Q$ is convex if and only if the line segment joining any two point $x, y \in Q$ lies entirely in $Q$. Throughout this section, we let $U$ be an open, convex subset of $\mathbf{E}$.

Consider a $C^1$-smooth function $f \colon U \to \mathbf{R}$ and a point $x \in U$. Classically, the linear function
$$l(x; y) = f(x) + \langle \nabla f(x), y - x\rangle$$
is a "best first-order approximation" of $f$ near $x$. If $f$ is $C^2$-smooth, then the quadratic function
$$Q(x; y) = f(x) + \langle \nabla f(x), y - x\rangle + \tfrac{1}{2}\langle \nabla^2 f(x)(y-x), y-x\rangle$$
is a "best second-order approximation" of $f$ near $x$. These two functions play a fundamental role when designing and analyzing algorithms, they furnish simple linear and quadratic local models of $f$. In this section, we aim to quantify how closely $l(x;\cdot)$ and $Q(x;\cdot)$ approximate $f$. All results will follow quickly by restricting multivariate functions to line segments and then applying the fundamental theorem of calculus for univariate functions. To this end, the following observation plays a basic role.

**Exercise 1.14.** ✍ Consider a function $f \colon U \to \mathbf{R}$ and two points $x, y \in U$. Define the univariate function $\varphi \colon [0,1] \to \mathbf{R}$ given by $\varphi(t) = f(x + t(y-x))$ and let $x_t := x + t(y-x)$ for any $t$.

1. Show that if $f$ is $C^1$-smooth, then equality
$$\varphi'(t) = \langle \nabla f(x_t), y - x\rangle \quad \text{holds for any } t \in (0,1).$$
2. Show that if $f$ is $C^2$-smooth, then equality
$$\varphi''(t) = \langle \nabla^2 f(x_t)(y-x), y-x\rangle \quad \text{holds for any } t \in (0,1).$$

The fundamental theorem of calculus now takes the following form.

**Theorem 1.15** (Fundamental theorem of multivariate calculus)**.** *Consider a $C^1$-smooth function $f\colon U \to \mathbf{R}$ and two points $x, y \in U$. Then equality*

$$f(y) - f(x) = \int_0^1 \langle \nabla f(x + t(y - x)), y - x\rangle \, dt,$$

*holds.*

*Proof.* Define the univariate function $\varphi(t) = f(x+t(y-x))$. The fundamental theorem of calculus yields the relation $\varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)\, dt$. Taking into account Exercise 1.14 completes the proof. □

The following corollary precisely quantifies the gap between $f(y)$ and its linear and quadratic models, $l(x; y)$ and $Q(x; y)$.

**Corollary 1.16** (Accuracy in approximation)**.** *Consider a $C^1$-smooth function $f\colon U \to \mathbf{R}$ and two points $x, y \in U$. Then we have*

$$f(y) = l(x; y) + \int_0^1 \langle \nabla f(x + t(y - x)) - \nabla f(x), y - x\rangle \, dt.$$

*If $f$ is $C^2$-smooth, then the equation holds:*

$$f(y) = Q(x; y) + \int_0^1 \int_0^t \langle (\nabla^2 f(x + s(y - x)) - \nabla^2 f(x))(y - x), y - x\rangle \, ds \, dt.$$

*Proof.* The first equation is immediate from Theorem 1.15. To see the second equation, define the function $\varphi(t) := f(x + t(y - x))$. Then applying the fundamental theorem of calculus twice yields

$$\begin{aligned} \varphi(1) - \varphi(0) &= \int_0^1 \varphi'(t)\, dt = \int_0^1 (\varphi'(0) + \int_0^t \varphi''(s)\, ds)\, dt \\ &= \varphi'(0) + \frac{1}{2}\varphi''(0) + \int_0^1 \int_0^t \varphi''(s) - \varphi''(0)\, ds\, dt. \end{aligned}$$

Appealing to Excercise 1.14, the result follows. □

Recall that if $f$ is differentiable at $x$, then the relation holds:

$$\lim_{y \to x} \frac{f(y) - l(x; y)}{\|y - x\|} = 0.$$

Indeed, this is the very definition of differentiability. An immediate consequence of Corollary 1.16 is that if $f$ is $C^1$-smooth, then the equation above is stable under perturbations of the base point $x$.

**Corollary 1.17** (First and second order expansions)**.** *Suppose that $f\colon U \to \mathbf{R}$ is $C^1$-smooth. Then for any point $\bar{x} \in U$, the estimate holds:*

$$\lim_{y \to x} \frac{f(y) - l(x;y)}{\|y - x\|} = 0.$$

*If $f$ is in addition $C^2$-smooth, then the estimate holds:*

$$\lim_{x,y \to \bar{x}} \frac{f(y) - Q(x;y)}{\|y - x\|^2} = 0. \tag{1.2}$$

When the mappings $\nabla f$ and $\nabla^2 f$ are Lipschitz continuous, one has even greater control on the accuracy of approximation, in essence passing from little-o terms to big-O terms.

**Exercise 1.18** (Accuracy in approximation under Lipschitz conditions)**.** Suppose $f\colon U \to \mathbf{R}$ is a $\beta$-smooth function. Then for any points $x, y \in U$ the inequality

$$\Big| f(y) - l(x;y) \Big| \le \frac{\beta}{2}\|y - x\|^2 \quad \text{holds.}$$

Moreover, if $f$ is $C^2$-smooth and satisfies the estimate

$$\|\nabla^2 f(y) - \nabla^2 f(x)\|_{\mathrm{op}} \le M\|y - x\| \qquad \text{for all } x, y \in U,$$

then the inequality

$$\Big| f(y) - Q(x;y) \Big| \le \frac{M}{6}\|y - x\|^3, \qquad \text{holds for all } x, y \in U.$$

[**Hint:** This follows directly from Corollary 1.16.]

Corollary 1.17 and Exercise 1.18 play central roles in optimization, as will become clear in later chapters. We end this section with one useful consequence of Corollary 1.17: derivative-based conditions for a point to be a local minimizer of a smooth function.

A point $x$ is called a *local minimizer* of a function $f\colon \mathbf{E} \to \overline{\mathbf{R}}$ if there exists a convex neighborhood $U$ of $x$ such that $f(x) \le f(y)$ for all $y \in U$. Observe that naively checking if $x$ is a local minimizer of $f$ from the very definition requires evaluation of $f$ at every point near $x$, an impossible task. We now derive a *verifiable necessary condition* for local optimality based on the gradient.

**Theorem 1.19.** *(First-order necessary conditions) Suppose that $x$ is a local minimizer of a function $f : U \to \mathbf{R}$. If $f$ is differentiable at $x$, then equality $\nabla f(x) = 0$ holds.*

*Proof.* Set $v := -\nabla f(x)$. Then for all small $t > 0$, the definition of differentiability implies

$$0 \leq \frac{f(x+tv) - f(x)}{t} = -\|\nabla f(x)\|^2 + \frac{o(t)}{t}.$$

Letting $t$ tend to zero yields $\nabla f(x) = 0$, as claimed. $\square$

To obtain *verifiable sufficient conditions* for optimality, higher order derivatives are required.

**Theorem 1.20.** *(Second-order conditions)*
*Consider a $C^2$-smooth function $f\colon U \to \mathbf{R}$ and fix a point $x \in U$. Then the following are true.*

1. *(Necessary conditions) If $x \in U$ is a local minimizer of $f$, then*

$$\nabla f(x) = 0 \quad \text{and} \quad \nabla^2 f(x) \succeq 0.$$

2. *(Sufficient conditions) If the relations*

$$\nabla f(x) = 0 \quad \text{and} \quad \nabla^2 f(x) \succ 0$$

*hold, then $x$ is a local minimizer of $f$. More precisely, it holds:*

$$\liminf_{y \to x} \frac{f(y) - f(x)}{\frac{1}{2}\|y - x\|^2} \geq \lambda_n(\nabla^2 f(x)).$$

*Proof.* Suppose first that $x$ is a local minimizer of $f$. Then Theorem 1.19 guarantees $\nabla f(x) = 0$. Consider an arbitrary vector $v \in \mathbf{E}$. Then for all small $t > 0$, we deduce from a second-order expansion (1.2) the estimate

$$0 \leq \frac{f(x+tv) - f(x)}{\frac{1}{2}t^2} = \langle \nabla^2 f(x) v, v\rangle + \frac{o(t^2)}{t^2}.$$

Letting $t$ tend to zero yields $\langle \nabla^2 f(x) v, v\rangle \geq 0$ for all $v \in \mathbf{E}$, as claimed.

Suppose $\nabla f(x) = 0$ and $\nabla^2 f(x) \succ 0$. Let $\epsilon > 0$ be such that $B_\epsilon(x) \subset U$. Then for points $y$ sufficiently close to $x$, the second-order expansion (1.2) yields the estimate

$$\begin{aligned}\frac{f(y) - f(x)}{\frac{1}{2}\|y - x\|^2} &= \left\langle \nabla^2 f(x) \left(\frac{y - x}{\|y - x\|}\right), \frac{y - x}{\|y - x\|} \right\rangle + \frac{o(\|y - x\|^2)}{\|y - x\|^2} \\ &\geq \lambda_n(\nabla^2 f(x)) + \frac{o(\|y - x\|^2)}{\|y - x\|^2}.\end{aligned}$$

Letting $y$ tend to $x$, the result follows. $\square$

The reader may be misled into believing that the role of the necessary conditions and the sufficient conditions for optimality (Theorem 1.20) is merely to determine whether a point $x$ is a local minimizer of a smooth function $f$. Such a viewpoint is far too limited.

Necessary conditions serve as the basis for algorithm design. If necessary conditions for optimality fail at a point, then there must be some point nearby with a strictly smaller objective value. A method for discovering such a point is a first step for designing algorithms. Sufficient conditions play an entirely different role. In later chapters, we will later see that sufficient conditions for optimality at a point $x$ guarantee that the function $f$ is *strongly convex* on a neighborhood of $x$. Strong convexity, in turn, is essential for establishing rapid convergence of numerical methods.

## Comments

All results in this chapter can be found in standard textbooks in linear algebra and real analysis. For more details on the material in Sections 1.1-1.3, the reader may refer to the relevant sections Boyd-Vandenberghe [4], Halmos [6], and Strang [14]. The details of Section 1.5 can be found in Rudin [13]. The content of Sections 1.6-1.7 can be found in most advanced calculus textbooks, such as Apostol [1] and Folland [5].

# Chapter 2

# Convex geometry

This chapter introduces the basic geometric and topological properties of convex sets. The material presented here will, in turn, serve as the foundation for convex analysis developed in Chapter **??**. The main goal for the reader should be to not only learn the formal theorems but to also develop intuition about convexity.

**Roadmap.** The chapter begins with Section 2.1 which recalls the definition of convex sets, introduces a few basic examples, and shows that convexity is preserved under various operations on sets, such as sums, intersections, and images/preimages by linear maps. Section 2.2 introduces the convex hull operation that associates to any set the smallest convex set that contains it. Section 2.3 discusses topological properties of convex sets. The key theorem proved in the section is that any convex set always has nonempty interior relative to the smallest affine space that contains it. Section 2.4 for the first time discusses the idea of hyperplane separation and duality. The main result is that any nonempty, closed, convex set admits a "dual description" as the intersection of all halfspaces containing it. An important construction motivated by such dual descriptions is the polar of a convex cone, discussed in Section 2.5. The final Section 2.6 introduces the cones of tangent and outward normal directions, which will play a central role in Chapter **??**.

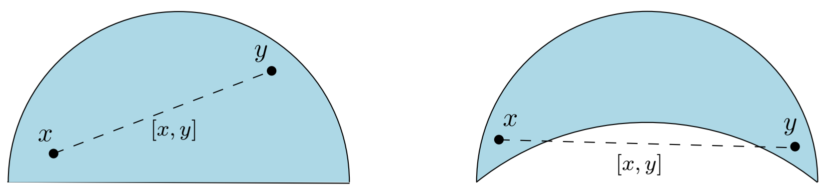
## 2.1 Operations preserving convexity

We begin with some convenient notation. For any two points $x$ and $y$ in $\mathbf{E}$, define the *closed line segment*

$$[x, y] := \{\lambda x + (1-\lambda)y \ : \ 0 \le \lambda \le 1\}.$$

The open line segment $(x, y)$ and the half-closed segments $[x, y)$ and $(x, y]$ are defined analogously. We have already encountered convex sets briefly in Chapter 1. Since convex set are the central objects of the current chapter, let us recall their defining property here.

**Definition 2.1** (Convex sets)**.** A set $Q \subseteq \mathbf{E}$ is said to be *convex* if for any two points $x, y \in Q$, the entire line segment $[x, y]$ is contained in $Q$.




(a) Convex (b) Not convex.

Figure 2.1: Convexity.

Let us look at a few basic examples. First, it is immediate from the definition that linear subspaces are convex. More generally, a set $L \subset \mathbf{E}$ is called *affine* if it is a translate of a linear subspace. In other words $L$ is affine if it has the form $L = v + S$ for some vector $v \in \mathbf{E}$ and a linear subspace $S \subset \mathbf{E}$. Since convexity is clearly preserved under translation, affine sets are convex. More interestingly, sets of the form $Q = \{x : \langle a, x\rangle \le b\}$, for some $a \in \mathbf{E}$ and $b \in \mathbf{R}$, are convex. Such sets are called *half-spaces*. A quick computation also shows that unit balls of arbitrary norms are convex sets; see Figure 1.1 for an illustration. The reader should verify that the nonnegative orthant

$$\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x \ge 0\}$$

and the cone of positive semi-definite matrices

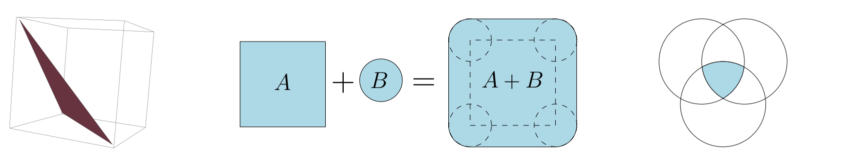
$$\mathbf{S}^n_+ = \{x \in \mathbf{S}^n : X \succeq 0\}$$

are convex. Here, the symbol "$\ge$" should be understood coordinatewise.

We thus have built a small (so far) library of convex sets. Verifying convexity from the definition is tedious and can often be avoided. The simplest way to argue that a set is convex is to recognize it as having been constructed from known convex sets (in our library) by a sequence of set operations that preserve convexity. In this section, we describe a few such convexity-preserving set operations. Refer to Section 1.4 for the sum, scaling, and image/preimage notation.

**Exercise 2.2** (Preservation of convexity). ✍ Prove the following statements.

1. **(Scaling)** For any convex set $A \subset \mathbf{E}$, the set $\mathbf{R}_+ A$ is convex.
2. **(Set addition)** For any two convex sets $Q_1, Q_2 \subset \mathbf{E}$, the sum $Q_1 + Q_2$ is convex. See Figure 2.4b for an example.
3. **(Intersection)** The intersection $\bigcap_{i \in I} Q_i$ of convex sets $Q_i \subset \mathbf{E}$, indexed by an arbitrary set $I$, is convex. See Figure 2.2c for an example.
4. **(Linear image/preimage)** For any convex sets $Q \subset \mathbf{E}$ and $L \in \mathbf{Y}$ and a linear map $\mathcal{A} \colon \mathbf{E} \to \mathbf{Y}$, the image $\mathcal{A}Q$ and the preimage $\mathcal{A}^{-1}L$ are convex sets.




(a) Simplex in $\mathbf{R}^3$. (b) Disk plus square. (c) Intersection of disks

Figure 2.2: Illustrations of convex sets.

Let us look now at two notable examples of sets built from convexity preserving operations. A *polyhedron* is any set of the form

$$Q = \{x \in \mathbf{R}^n : Ax \geq c\},$$

for some $A \in \mathbf{R}^{m \times n}$ and $c \in \mathbf{R}^m$. Equivalently, we may write $Q$ as an intersection of finitely many halfspaces or as the preimage $A^{-1}(c + \mathbf{R}^n_+)$. Appealing to Exercise (2.2), we deduce that polyhedra are convex. Linear programming refers to the problem of minimizing a linear function over a polyhedron.

More generally, a *spectrahedron* is any set of the form

$$Q = \{x \in \mathbf{R}^n : x_1 A_1 + x_2 A_2 + \ldots + x_n A_n \succeq C\},$$

for some matrices $A_i \in \mathbf{S}^m$ and $C \in \mathbf{S}^n$. Equivalently, we may write $Q$ as the preimage $\mathcal{A}^{-1}(C + \mathbf{S}^n_+)$ for the linear map $\mathcal{A}(x) = \sum_{i=1}^n x_i A_i$. Appealing to Exercise (2.2), we deduce that spectrahedra are convex. Semidefinite programming refers to the problem of minimizing a linear function over a spectrahedron.

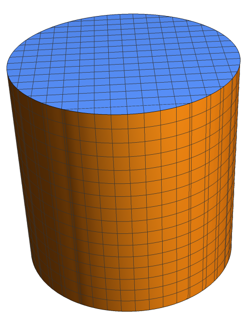
There are many more spectrohedra than polyhedra. For example, a quick computation shows that a cylinder can be written as the spectrahedron (see Figure 2.3a):

$$\left\{ (x,y,z) \in \mathbf{R}^3 : \begin{pmatrix} 1+x & y & 0 & 0 \\ y & 1-x & 0 & 0 \\ 0 & 0 & 1+z & 0 \\ 0 & 0 & 0 & 1-z \end{pmatrix} \succeq 0 \right\}.$$

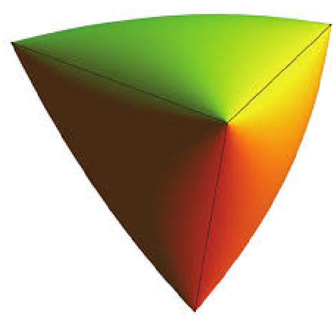
A more interesting example, depicted in Figure 2.3b is the elliptope:

$$\left\{ (x,y,z) \in \mathbf{R}^3 : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\}$$

The high dimensional version of this set appears in statistics as the set of correlation matrices and in combinatorial optimization when forming convex relaxations of NP-hard problems.



(a) Cylinder.



(b) The elliptope

Figure 2.3: Spectrahedra.

It will be important for the sets that we encounter to not only be convex but to also be closed. Not all set operations in Exercise 2.2 preserve closed sets. Whereas intersections and linear preimages of closed sets are closed (why?), sums and linear images of closed sets need not be closed in general.

The following exercise presents two closed convex sets in $\mathbf{R}^3$ whose sum is not closed. Similarly, the image of a closed set under a linear map may also fail to be closed (why?). Though this pathology may seem like a technicality, it can have pronounced negative consequences; e.g. strong duality failing in convex optimization. Therefore, care must be taken when dealing with closure issues.

**Exercise 2.3.** Do the following exercises.

1. Show that if $Q_1, Q_2 \subset \mathbf{E}$ are closed sets and $Q_1$ is bounded, then the sets $\mathbf{R}_+ Q_1$ and $Q_1 + Q_2$ are closed.
2. Give an example of a closed set $Q \in \mathbf{R}^2$ such that $\mathbf{R}_+ Q$ is not closed.
3. Define the two closed sets

   $$Q_1 = \{(x, y, r) \in \mathbf{R}^3 : \sqrt{x^2 + y^2} \leq r\} \qquad \text{and} \qquad Q_2 = \{(0, \lambda, \lambda) : \lambda \geq 0\}.$$

   Show that the sum $Q_1 + Q_2$ is not a closed set.

## 2.2 Convex hull

The notion of a linear combination of vectors plays a central role in linear algebra. Convex combinations of points play a similarly central role in convex geometry. To simplify notation, define the *unit simplex* (see Figure 2.2a)

$$\Delta_n := \left\{ \lambda \in \mathbf{R}^n : \sum_{i=1}^{n} \lambda_i = 1, \lambda \geq 0 \right\}.$$

**Definition 2.4** (Convex combination)**.** A point $x \in \mathbf{E}$ is a *convex combination* of points $x_1, \ldots, x_k \in \mathbf{E}$ if it can be written as $x = \sum_{i=1}^{k} \lambda_i x_i$ for some $\lambda \in \Delta_k$.

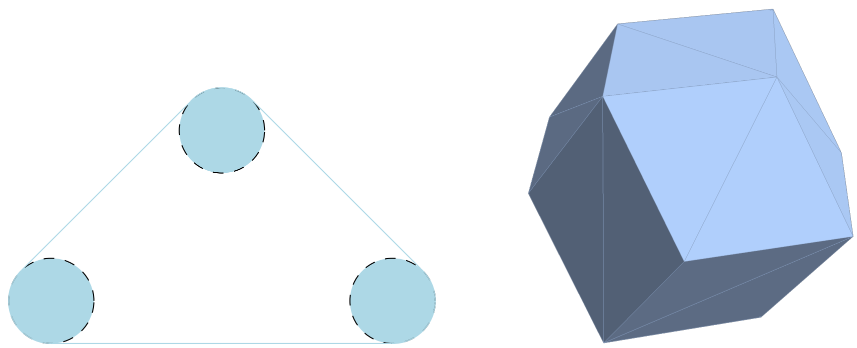
A good way to think about a representation of $x$ as a convex combination $x = \sum_{i=1}^{k} \lambda_i x_i$ is to regard $x$ as a weighted average of $x_1, \ldots, x_k$ with $\lambda_1, \ldots, \lambda_k$ as the corresponding weights. Observe that convexity of a set $Q \subset \mathbf{E}$ guarantees that convex combinations of any two points of $Q$ lie in $Q$; indeed, this property defines convexity. The following exercise shows that convexity of $Q$ entails a seemingly stronger property: convex combinations of any finite number of points of $Q$ lie in $Q$.

**Exercise 2.5.** ✍ Consider a convex set $Q \subset \mathbf{E}$ and let $k \in \mathbb{N}$ be arbitrary. Show that any convex combination of points $x_1, \ldots, x_k \in Q$ lies in $Q$.
[**Hint:** Rewrite $\sum_{i=1}^{k} \lambda_i x_i = (1 - \lambda_k) \sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k} x_i + \lambda_k x_k$ and reason inductively.]

For any nonconvex set $Q$, one can imagine forming the "minimal" convex set that contains $Q$. The resulting convex set is called the convex hull of $Q$.

**Definition 2.6** (Convex hull)**.** The *convex hull* of a set $Q \subseteq \mathbf{E}$, denoted $\operatorname{conv}(Q)$, is the intersection of all convex sets containing $Q$.

Notice that by Exercise 2.2, the convex hull $\operatorname{conv}(Q)$ is a convex set. One can visualize the convex hull of a set $Q \subset \mathbf{R}^2$ by encircling $Q$ with a rubber band and letting it contract. The outline of the rubber band marks the boundary of the convex hull. See Figure 2.4 for an illustration.



(a) Convex hull of three disks (b) $\operatorname{conv}\{(\pm 1, \pm 1, \pm 1), \pm 2e_1, \pm 2e_2, \pm 2e_3)\}$.

Figure 2.4: Convex hull.

The definition of the convex hull of $Q$ is external: it involves sets that are larger than $Q$. The following exercise provides an equivalent internal description of $\operatorname{conv}(Q)$ as the set of all convex combinations of points in $Q$.

**Exercise 2.7.** ✍ For any set $Q \subset \mathbf{E}$, prove the equality:

$$\operatorname{conv}(Q) = \left\{ \sum_{i=1}^{k} \lambda_i x_i \ : \ k \in \mathbb{N},\ x_1, \ldots, x_k \in Q,\ \lambda \in \Delta_k \right\}. \tag{2.1}$$

The description (2.1) does not rule out that one might have to take $k$ arbitrarily large in order to obtain the entire convex hull $\operatorname{conv}(Q)$. On the contrary, the following theorem shows that it suffices to take $k \leq n + 1$, where $n$ is the dimension of $\mathbf{E}$.

**Theorem 2.8** (Carathéodory)**.** *Consider a set $Q \subset \mathbf{E}$, where $\mathbf{E}$ is an $n$-dimensional Euclidean space. Then each point $x \in \operatorname{conv}(Q)$ can be written as a convex combination of at most $n + 1$ points in $Q$.*

*Proof.* Since $x$ belongs to $\mathrm{conv}(Q)$, we may write $x = \sum_{i=1}^{k} \lambda_i x_i$ for some integer $k$, points $x_1, \ldots, x_k \in Q$, and weights $\lambda \in \Delta_k$. We may assume $k \geq n+2$, since otherwise there is nothing to prove. We claim that we may rewrite $x$ as a convex combination of at most $k-1$ points.

We begin the argument by noticing that the vectors

$$x_2 - x_1, \ldots, x_k - x_1$$

are linearly dependent, since there are at least $n-1$ of them. Therefore, there exist numbers $\mu_i$ for $i = 2, \ldots, k$ not all zero and satisfying $0 = \sum_{i=2}^{k} \mu_i(x_i - x_1) = \sum_{i=2}^{k} \mu_i x_i - (\sum_{i=2}^{k} \mu_i) x_1$. Defining $\mu_1 := -\sum_{i=2}^{k} \mu_i$, we deduce $\sum_{i=1}^{k} \mu_i x_i = 0$ and $\sum_{i=1}^{k} \mu_i = 0$. Then for any $\alpha \in \mathbf{R}$, we compute

$$x = \sum_{i=1}^{k} \lambda_i x_i - \alpha \sum_{i=1}^{k} \mu_i x_i = \sum_{i=1}^{k} (\lambda_i - \alpha\mu_i) x_i$$

and

$$\sum_{i=1}^{k} (\lambda_i - \alpha\mu_i) = 1.$$

We will now choose $\alpha$ so that all the coefficients $\lambda_i - \alpha\mu_i$ are nonnegative and at least one of them is zero. To this end, observe that since the vector $\mu$ is not zero, it has at least one positive coordinate. Therefore, we may choose an index $i^* \in \mathrm{argmin}_i\{\lambda_i/\mu_i : \mu_i > 0\}$ and set $\alpha = \frac{\lambda_{i^*}}{\mu_{i^*}}$. Thus $x$ is a convex combination of $k-1$ points, as the coefficient $\lambda_{i^*} - \alpha\mu_{i^*}$ is zero. Continuing this process, we will obtain a description of $x$ as a convex combination of $k \leq n+1$ points. □

## 2.3 Affine hull and relative interior

Convex sets can easily have empty interior. For example, the unit simplex $\Delta_n$ has empty interior in its ambient space $\mathbf{R}^n$. The main result of this section shows that any nonempty convex set $Q$ has nonempty interior "relative" to the smallest affine space that contains it. The main use of the relative interior in later sections will be to show that convex functions are very well-behaved within the relative interior of their domains.

Recall that a set is called affine if it is a translate if it has the form $L = v + S$ for some vector $v \in \mathbf{E}$ and a linear subspace $S \subset \mathbf{E}$. In particular, affine sets that contain the origin are linear subspaces (why?).

**Definition 2.9** (Affine hull)**.** The *affine hull* of a set $Q \subset \mathbf{E}$, denoted by $\operatorname{aff}(Q)$, is the intersection of all affine sets that contain $Q$.

It is straightforward to check that $\operatorname{aff} Q$ is itself an affine set, and is by definition the smallest affine set that contains $Q$. See Figure 2.5 for an illustration. For example, the affine hull of the unit simplex $\Delta_n$ is the hyperplane $\{(x, y, z) : x+y+z = 1\}$. The reader should convince themselves that if $Q$ contains the origin, then $\operatorname{aff} Q$ coincides with the linear span of $Q$.

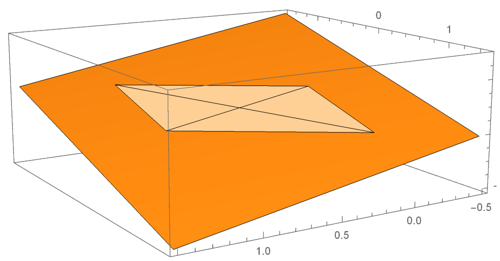



Figure 2.5: The convex set $\operatorname{conv}\{(e_1, e_2, e_3, ((1, -1, 1))\}$ and its affine hull.

Viewing $\operatorname{aff} Q$ as the ambient space of $Q$, it is appealing to focus on the interior of $Q$ relative to this smaller ambient space.

**Definition 2.10** (Relative interior and boundary)**.** The *relative interior* of a set $Q \subset \mathbf{E}$, denoted $\operatorname{ri} Q$, is the interior of $Q$ relative to $\operatorname{aff}(Q)$. That is, we set

$$\operatorname{ri} Q := \{x \in Q : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \cap \operatorname{aff}(Q) \subseteq Q\}.$$

The *relative boundary* of $Q$ is defined by $\operatorname{rb} Q := Q \setminus (\operatorname{ri} Q)$.

The following is the main result of the section.

**Theorem 2.11** (Relative interior is nonempty)**.** *For any nonempty convex set $Q \subset \mathbf{E}$, the relative interior $\operatorname{ri} Q$ is nonempty.*

*Proof.* Without loss of generality, we may translate $Q$ to contain the origin. Then $\operatorname{aff} Q$ contains the origin and is therefore a linear subspace. Let $d$ be the dimension of $\operatorname{aff} Q$ as a linear subspace. Observe that $Q$ must contain some $d$ linearly independent vectors $x_1, \ldots, x_d$, since otherwise $\operatorname{aff} Q$ would have a smaller dimension than $d$. Define the linear map $A \colon \mathbf{R}^d \to \operatorname{aff} Q$ by $A(\lambda_1, \ldots, \lambda_d) = \sum_{i=1}^d \lambda_i x_i$. Since the range of $A$ contains $x_1, \ldots, x_d$, the map $A$ is surjective. Hence $A$ is a linear isomorphism. Consequently $A$ maps the open set

$$\Omega := \left\{ \lambda \in \mathbf{R}^d : \sum_{i=1}^d \lambda_i < 1 \text{ and } \lambda_i > 0 \text{ for all } i \right\}$$

to an open subset $A(\Omega)$ of $\operatorname{aff} Q$. Note for any $\lambda \in \Omega$, we can write $A\lambda = \sum_{i=1}^{d} \lambda_i x_i + (1 - \sum_{i=1}^{d} \lambda_i) \cdot 0$. Hence, convexity of $Q$ implies $A(\Omega) \subset Q$, thereby proving $\operatorname{ri} Q \neq \emptyset$. □

**Exercise 2.12.** ✍ Show that for any convex set $Q \subset \mathbf{E}$, the two sets, $\operatorname{cl} Q$ and $\operatorname{ri} Q$, are convex and have the same affine hull as $Q$ itself.

One important consequence of (2.11) is that any closed convex set coincides with the closure of its relative interior. This result, proved in Corollary 2.14, will follow quickly from the following lemma.

**Theorem 2.13** (Accessibility)**.** *Consider a convex set $Q$ and two points $x \in \operatorname{ri} Q$ and $y \in \operatorname{cl} Q$. Then the line segment $[x, y)$ is contained in $\operatorname{ri} Q$.*

*Proof.* Without loss of generality, suppose that the affine hull of $Q$ is all of $\mathbf{E}$. Then since $x$ lies in the interior of $Q$, there exists $\epsilon > 0$ satisfying $B_\epsilon(x) \subset Q$. Define the open set $\Lambda := \{\lambda z + (1-\lambda)y : z \in B_\epsilon(x), \lambda \in (0,1)\}$; see Figure 2.6. Since $Q$ is convex, the inclusion $[x, y) \subset \Lambda \subset Q$ holds. □

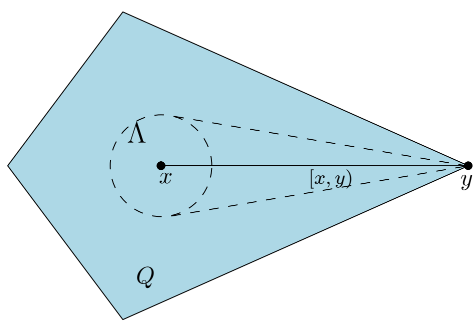



Figure 2.6: Illustration of the proof of Theorem 2.13

**Corollary 2.14.** *For any nonempty convex set $Q$ in $\mathbf{E}$, equalities holds:*

$$\operatorname{cl}(\operatorname{ri} Q) = \operatorname{cl} Q \quad \text{and} \quad \operatorname{ri}(\operatorname{cl} Q)) = \operatorname{ri} Q.$$

*Proof.* We begin by verifying the first equation. The inclusion $\operatorname{ri} Q \subseteq Q$ immediately implies $\operatorname{cl}(\operatorname{ri} Q) \subseteq \operatorname{cl} Q$. Conversely, fix a point $y \in \operatorname{cl} Q$. Since $\operatorname{ri} Q$ is nonempty by Theorem 2.11, we may also choose a point $x \in \operatorname{ri} Q$. Theorem 2.13 then guarantees $y \in \operatorname{cl}[x, y) \subseteq \operatorname{cl}(\operatorname{ri} Q)$. Since the point $y \in \operatorname{cl} Q$ is arbitrary, we have established the equality $\operatorname{cl}(\operatorname{ri} Q) = \operatorname{cl} Q$.

Next, we verify the second equation. Without loss of generality, we may suppose that $Q$ contains the origin and therefore that $\operatorname{aff}(Q)$ is a linear subspace. The inclusion $\supset$ is clear. Fix any point $z \in \operatorname{ri}(\operatorname{cl} Q)$ and choose an arbitrary point $x \in \operatorname{ri} Q$. We may assume $x \neq z$, since otherwise the inclusion $z \in \operatorname{ri} Q$ would hold trivially. Observe the equality $\operatorname{aff} Q = \operatorname{aff}(\operatorname{cl} Q)$ (verify this!). Fix a constant $\mu > 0$ and define the point

$$y := z + \mu(z - x).$$

Since $\operatorname{aff} Q$ is a linear subspace, the inclusion $y \in \operatorname{aff} Q$ holds. Therefore the definition of the relative interior guarantees $y \in \operatorname{cl} Q$ for all sufficiently small $\mu > 0$. Rearranging the equation, we deduce

$$z = \frac{1}{\mu} y + \frac{\mu}{1+\mu} x \in (y, x).$$

Thus by Theorem 2.13, the inclusion $z \in \operatorname{ri} Q$ holds. □

## 2.4 Separation theorem

One of the most fruitful ways to study properties of sets is to instead focus on the functions that act on them. This is the principle of duality. This section introduces duality within the context of convex geometry. The main result is the principle of strict separation: any point $y$ lying outside a closed, convex set $Q$ can be separated from $Q$ by a hyperplane. An important consequence is the dual description of convex sets. Tautologically a convex set $Q$ is simply a collection of points. On the other hand, we will see that $Q$ coincides with the intersection of all half-spaces containing it.

We begin with the following basic definitions. Along with any set $Q \subset \mathbf{E}$ we define the *distance function*

$$\operatorname{dist}_Q(y) := \inf_{x \in Q} \|x - y\|,$$

and the *projection*

$$\operatorname{proj}_Q(y) := \{x \in Q : \operatorname{dist}_Q(y) = \|x - y\|\}.$$

Thus $\operatorname{proj}_Q(y)$ consists of all the nearest points of $Q$ to $y$; see Figure 2.7 for an illustration.

**Exercise 2.15.** ✍ Show that for any nonempty set $Q \subseteq \mathbf{E}$, the function $\operatorname{dist}_Q \colon \mathbf{E} \to \mathbf{R}$ is 1-Lipschitz.

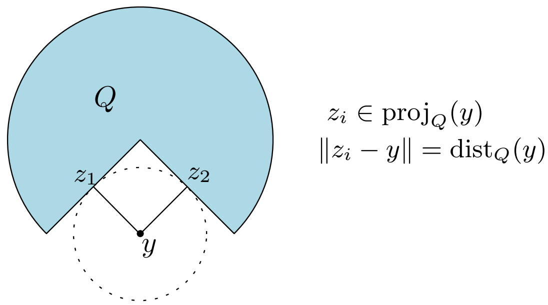



Figure 2.7: Nearest-point projection

If $Q$ is closed, then the nearest-point set $\mathrm{proj}_Q(y)$ is nonempty for any $y \in \mathbf{E}$. To see this, fix a point $y \in Q$ and define the function

$$\varphi(x) = \begin{cases} \|x - y\| & \text{if } x \in Q \\ +\infty & \text{if } x \notin Q \end{cases}.$$

The set of minimizers of $\varphi$ coincides with $\mathrm{proj}_Q(y)$. Since $\varphi$ is closed and coercive, Theorem 1.9 guarantees that $\varphi$ has at least one minimizer, and therefore $\mathrm{proj}_Q(y)$ is nonempty.

The following theorem shows that when $Q$ is closed and convex, the set $\mathrm{proj}_Q(y)$ is not only nonempty, but is also a singleton. Moreover, the nearest-point $z \in Q$ to $y$ is characterized by the fact that the vector $y - z$ makes an obtuse angle with the vector $x - z$ for any $x \in Q$. See Figure 2.8 for an illustration.

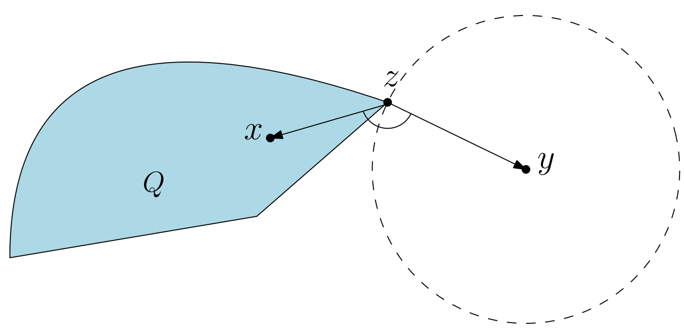



Figure 2.8: Nearest-point projection for convex sets

**Theorem 2.16** (Properties of the projection)**.** *For any nonempty, closed, convex set $Q \subset \mathbf{E}$, the set $\mathrm{proj}_Q(y)$ is a singleton. Moreover, the closest* *point $z \in Q$ to $y$ is characterized by the property:*

$$\langle y - z, x - z\rangle \leq 0 \qquad \text{for all } x \in Q. \tag{2.2}$$

*Proof.* Fix a point $y \notin Q$. The claim that any point $z$ satisfying (2.2) lies in $\mathrm{proj}_Q(y)$ is an easy exercise (verify it!). We therefore prove the converse. To this end, fix a point $z \in \mathrm{proj}_Q(y)$ and an arbitrary $x \in Q$. For each $t \in [0,1]$, define

$$x_t := x + t(y - x) \qquad \text{and} \qquad \varphi(t) := \tfrac{1}{2}\|y - x_t\|^2.$$

Convexity implies $x_t \in Q$ for all $t \in [0,1]$ and therefore

$$\varphi(t) \geq \tfrac{1}{2}\mathrm{dist}^2_Q(y) = \varphi(0).$$

Taking the derivative of $\varphi$, we therefore deduce

$$0 \leq \lim_{t \searrow 0} \frac{\varphi(t) - \varphi(0)}{t} = \varphi'(0) = -\langle y - z, x - z\rangle,$$

as claimed. Thus, a points $z$ lies in $\mathrm{proj}_Q(y)$ if and only if (2.2) holds.

To see that $\mathrm{proj}_Q(y)$ is a singleton, consider any two points $z, z' \in \mathrm{proj}_Q(y)$. Then, the estimate (2.2) for $z$ and $z'$ (with $x = z'$ and $x = z$, respectively) becomes

$$\langle y - z, z' - z\rangle \leq 0 \qquad \text{and} \qquad \langle y - z', z - z'\rangle \leq 0.$$

Adding the two inequalities yields $0 \geq \langle z - z', z - z'\rangle = \|z - z'\|^2$, and therefore $z = z'$ as we had to show. □

**Exercise 2.17.** ✍ Show that for any nonempty, closed, convex set $Q \subset \mathbf{E}$, the map $x \mapsto \mathrm{proj}_Q(x)$ is 1-Lipschitz.

It is worthwhile to note that the converse of Theorem 2.16 is also true. Namely, a closed set $Q$ in $\mathbf{E}$ is convex if and only if $\mathrm{proj}_Q(y)$ is a singleton for every point $y \in \mathbf{E}$. We will not use this equivalence and therefore omit the proof.

Theorem 2.16 allows to quickly prove the following fundamental property of convex sets. Given any closed convex set $Q$ and a point $y \notin Q$, there exists a hyperplane that separates $y$ from $Q$. See Figure 2.9 for an illustration.

**Theorem 2.18** (Strict separation)**.** *Consider a nonempty, closed, convex set $Q \subset \mathbf{E}$ and a point $y \notin Q$. Then there exists a nonzero vector $a \in \mathbf{E}$ and a number $b \in \mathbf{R}$ satisfying*

$$\langle a, x\rangle \leq b < \langle a, y\rangle \quad \text{for all } x \in Q.$$

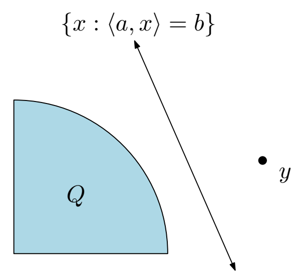



Figure 2.9: Basic separation

*Proof.* Fix a point $y \notin Q$ and define the nonzero vector $a := y - \mathrm{proj}_Q(y)$. Then for any $x \in Q$, the condition (2.2) yields

$$\langle a, x\rangle \leq \langle a, \mathrm{proj}_Q(y)\rangle = \langle a, y\rangle - \|a\|^2 < \langle a, y\rangle,$$

as claimed. □

**Exercise 2.19** (Supporting halfspaces)**.** Consider a convex set $Q \subset \mathbf{E}$. A halfspace $H \subset \mathbf{E}$ is said to *support* $Q$ *at a point* $x \in \mathrm{cl}\, Q$ if the inclusions, $x \in \mathrm{bd}\, H$ and $Q \subset H$, hold. Show that a convex set $Q$ admits a supporting halfspace at a point $x \in \mathrm{cl}\, Q$ if and only if $x$ lies on the boundary of $Q$. [**Hint:** Apply Theorem 2.18 with $y \notin Q$ tending to $x$.]

In particular, we can now establish the following "dual description" of convex sets, alluded to in the beginning of the section; see Figure 2.10.

**Exercise 2.20.** ✍ Given a nonempty set $Q \subset \mathbf{E}$, define the set of halfspaces

$$\mathcal{F}_Q := \{(a, b) \in \mathbf{E} \times \mathbf{R} : \langle a, x\rangle \leq b \quad \text{for all } x \in Q\}.$$

1. Show that $\mathcal{F}_Q$ is a convex subset of $\mathbf{E} \times \mathbf{R}$
2. Prove that $\mathcal{F}_Q$ is empty if and only if $\mathrm{cl}\,\mathrm{conv}(Q) = \mathbf{E}$.
3. Assuming $\mathcal{F}_Q$ is nonempty, show that equality holds:

$$\mathrm{cl}\,\mathrm{conv}(Q) = \bigcap_{(a,b)\in\mathcal{F}_Q} \{x \in \mathbf{E} : \langle a, x\rangle \leq b\}.$$

[**Hint:** Argue using strict separation (Theorem 2.18).]

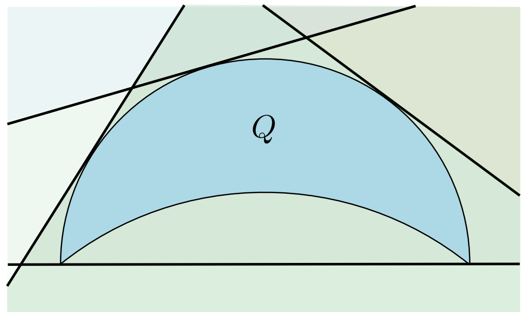



Figure 2.10: Closed convex envelope.

## 2.5 Cones and polarity

A particularly appealing class of sets consists of those that are invariant under scaling by nonnegative numbers.

**Definition 2.21** (Cones)**.** A set $K \subseteq \mathbf{E}$ is called a *cone* if the inclusion $\lambda K \subset K$ holds for any $\lambda \geq 0$.

For example, the nonnegative orthant $\mathbf{R}^n_+$ and the set of positive semidefinite matrices $\mathbf{S}^n_+$ are closed convex cones.

**Exercise 2.22.** ✍ Show that a set $K \subset \mathbf{E}$ is a convex cone if and only if the point $\lambda x + \mu y$ lies in $K$ for any two points $x, y \in K$ and numbers $\lambda, \mu \geq 0$.

**Exercise 2.23.** Prove for any convex cone $K \subset \mathbf{E}$ the equality

$$\operatorname{aff}(K) = K - K.$$

Duality ideas, explored in Exercise 2.20 simplify significantly for cones. Namely, consider a cone $K$ and a halfspace

$$H = \{x : \langle a, x\rangle \leq b\}$$

that contains it. Since $K$ contains the origin, $b$ is nonnegative. Moreover, taking into account positive homogeneity of $K$, the halfspace

$$\overline{H} = \{x : \langle a, x\rangle \leq 0\}, \tag{2.3}$$

provides a tighter approximation $Q \subset \overline{H} \subset H$. The set of all halfspaces of the form (2.3) that contain $K$ comprise the polar cone.

**Definition 2.24** (Polar cone)**.** The *polar cone* of a cone $K \subset \mathbf{E}$ is the set

$$K^\circ := \{v \in \mathbf{E} : \langle v, x\rangle \leq 0 \text{ for all } x \in K\}.$$

Thus $K^\circ$ consists of all vectors $v$ that make an obtuse angle with every vector $x \in K$. Observe that $K^\circ$ is always a closed convex cone since it is defined as the intersection of infinitely many half-spaces. See Figure 2.11 for an illustration.

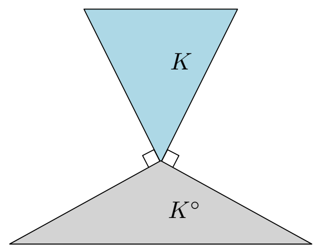



Figure 2.11: Polar cone

**Exercise 2.25.** Verify the following:

1. The polar cone of a linear subspace $L \subset \mathbf{E}$ is the orthogonal complement $L^\circ = L^\perp$.
2. $(\mathbf{R}^n_+)^\circ = \mathbf{R}^n_-$ and $(\mathbf{S}^n_+)^\circ = \mathbf{S}^n_-$

The following exercise shows a fundamental property of the polarity operation. Applying the polar operation twice to a cone $K$ yields its closed convex hull. The proof is immediate from Exercise 2.20.

**Exercise 2.26** (Double-polar theorem)**.** ✍ Prove for any nonempty cone $K \subset \mathbf{E}$ the equality:

$$(K^\circ)^\circ = \operatorname{cl}\operatorname{conv}(K).$$

Classically, the orthogonal complement to a sum of linear subspaces is the intersection of their orthogonal complements. In much the same way, the polarity operation satisfies "calculus rules".

**Theorem 2.27** (Polarity calculus)**.** *For any linear mapping* $\mathcal{A}\colon \mathbf{E} \to \mathbf{Y}$ *and a nonempty cone* $K \subset \mathbf{Y}$*, the chain rule holds*

$$(\mathcal{A}K)^\circ = (\mathcal{A}^*)^{-1}K^\circ. \tag{2.4}$$

*In particular, for any two nonempty cones* $K_1, K_2 \subset \mathbf{E}$*, the sum rule holds:*

$$(K_1 + K_2)^\circ = K_1^\circ \cap K_2^\circ \tag{2.5}$$

*Proof.* The definition of polarity yields the equivalences

$$\begin{aligned} y \in (\mathcal{A}K)^\circ &\iff \langle \mathcal{A}x, y\rangle \leq 0 \text{ for all } x \in K \\ &\iff \langle x, \mathcal{A}^* y\rangle \leq 0 \text{ for all } x \in K \\ &\iff \mathcal{A}^* y \in K^\circ \\ &\iff y \in (\mathcal{A}^*)^{-1} K^\circ. \end{aligned}$$

This establishes (2.4). The sum rule (2.5) follows from applying (2.4) to the expression $\mathcal{A}(K_1 \times K_2)$ with the mapping $\mathcal{A}(x, y) := x + y$. □

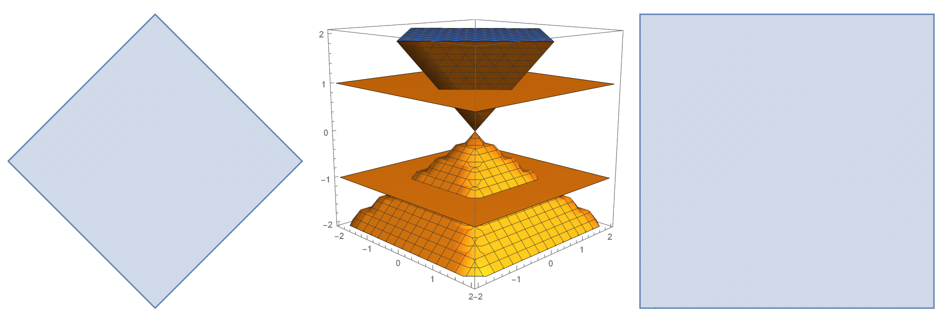
There is a convenient notion of polarity for general sets (not cones) based on "homogenizing" the set and then applying the polarity operation for cones. Consider a set $Q \subset \mathbf{E}$ and let $K$ be the cone generated by $Q \times \{1\} \subset \mathbf{E} \times \mathbf{R}$, that is

$$K = \{(\lambda x, \lambda) \in \mathbf{E} \times \mathbf{R} : x \in \mathbf{E}, \lambda \geq 0\}.$$

It is then natural to define the *polar set* as

$$Q^\circ := \{x \in \mathbf{E} : (x, -1) \in K^\circ\}.$$

The reader should refer to Figure 2.12 for an illustration.



(a) $Q = \{x : \|x\|_1 \leq 1\}$ (b) Homogenization (c) $Q^\circ = \{x : \|x\|_\infty \leq 1\}$

Figure 2.12: Figure 2.12a depicts $Q$, the unit ball of the $\ell_1$-norm. Figure 2.12b depicts the homogenization of $Q$, namely $K = \{(x, y, r) : r \geq |x| + |y|\}$ and the polar cone $K^\circ = \{(x, y, r) : r \leq -\max\{|x|, |y|\}\}$, along with the parallel hyperplanes $\mathbf{R}^2 \times \{\pm 1\}$. Figure 2.12c depicts $Q^\circ$, which can be identified with the intersection of $K^\circ$ with the hyperplane $\mathbf{R}^2 \times \{-1\}$.

Unraveling the definitions, the following algebraic description of the polar appears.

**Exercise 2.28.** ✍ Show for any set $Q \subset \mathbf{E}$, the equality

$$Q^\circ = \{v \in \mathbf{E} : \langle v, x\rangle \leq 1 \text{ for all } x \in Q\}.$$

The reader should convince themselves that if $Q$ happens to be a cone, than the above definition of the polar coincides with the definition of the polar we have given for cones. The following is a direct analogue of Theorem 2.26

**Exercise 2.29** (Double polar)**.** For any set $Q \subset \mathbf{E}$ containing the origin, we have

$$(Q^\circ)^\circ = \operatorname{cl}\operatorname{conv}(Q).$$

In particular, polarity of unit norm balls is in correspondence with duality of norms.

**Exercise 2.30** (Polarity and dual norms)**.** Let $\rho(\cdot)$ be an arbitrary norm on $\mathbf{E}$ and define its unit ball

$$B_\rho = \{x \in \mathbf{R} : \rho(x) \leq 1\}.$$

Show that the polar of $B_\rho$ is the unit ball of the dual norm:

$$B_\rho^\circ = B_{\rho^*}.$$

## 2.6 Tangents and normals

A principle technique in mathematical analysis is to reason about sets and functions using their first-order approximations. Both theory and algorithms rely on such approximations. This section revisits this idea by constructing first-order approximations of convex sets.

Consider a set $Q \subset \mathbf{E}$ and a point $\bar{x} \in Q$. Intuitively, we should think of a first-order approximation to $Q$ at $\bar{x}$ as the set of all limits of rays $\mathbf{R}_+(x_i - \bar{x})$ over all possible sequences $x_i \in Q$ tending to $\bar{x}$. With this intuition in mind, we introduce the following definition

**Definition 2.31** (Tangent cone)**.** The *tangent cone* to a set $Q \subset \mathbf{E}$ at a point $\bar{x} \in Q$ is the set

$$T_Q(\bar{x}) := \left\{\lim_{i\to\infty} \tau_i^{-1}(x_i - \bar{x}) : x_i \to \bar{x} \text{ in } Q,\ \tau_i \searrow 0\right\}.$$

In the definition, the sequence $\tau_i$ simply rescales the approach directions $x_i - \bar{x}$. See Figure 2.13 for an illustation. The reader should convince themselves that $T_Q(\bar{x})$ is a closed cone, which need not be convex in general. Whenever $Q$ is convex, the definition simplifies drastically.

**Exercise 2.32.** ✍ Show for any convex set $Q \subset \mathbf{E}$ and a point $\bar{x} \in Q$ the equality:

$$T_Q(\bar{x}) = \text{cl } \mathbf{R}_+(Q - \bar{x}).$$

[**Hint:** The inclusion $\subset$ is clear and does not use convexity. The reverse inclusion follows from the definition of convexity and the fact that $T_Q(\bar{x})$ is closed.]

Thus for a convex set, the tangent cone $T_Q(\bar{x})$ is computed by shifting $Q$ so that $\bar{x}$ becomes the origin and then taking the closure of all nonnegative scalings of the shifted set; see Figure 2.14.

Tangency concerns directions pointing "into the set". Alternatively, we can also think dually of outward normal vectors to a set $Q$ at $\bar{x} \in Q$. Geometrically, it is intuitive to call a vector $v$ an (outward) normal to $Q$ at $\bar{x}$ if $Q$ is fully contained in the half-space $\{x \in \mathbf{E} : \langle v, x - \bar{x}\rangle \leq 0\}$ up to a first-order error.

**Definition 2.33** (Normal cone)**.** The *normal cone* to a set $Q \subset \mathbf{E}$ at a point $\bar{x} \in Q$ is the set

$$N_Q(\bar{x}) := \{v \in \mathbf{E} : \langle v, x - \bar{x}\rangle \leq o(\|x - \bar{x}\|) \quad \text{as } x \to \bar{x} \text{ in } Q\}.$$

Thus a vector $v$ lies in $N_Q(\bar{x})$ if

$$\limsup_{x \to \bar{x}} \frac{\langle v, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0.$$

See Figure 2.13 for an illustration.

The following result shows that the normal cone $N_Q(\bar{x})$ is always the polar of the tangent cone $T_Q(\bar{x})$. In particular, $N_Q(\bar{x})$ is always a closed convex cone, even if $Q$ is not convex. Consequently, taking into account the double polar formula (Exercise 2.26), equality $T_Q(\bar{x}) = (N_Q(\bar{x}))^\circ$ holds if and only if $T_Q(\bar{x})$ is closed and convex.

**Lemma 2.34.** ✍ *For any set $Q \subset \mathbf{E}$ and a point $\bar{x} \in Q$, the polarity relationship holds:*

$$N_Q(\bar{x}) = (T_Q(\bar{x}))^\circ.$$

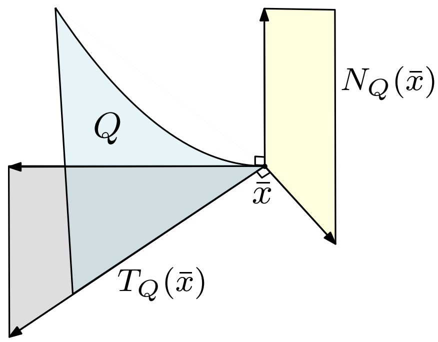



Figure 2.13: Illustration of the tangent and normal cones for nonconvex sets.

*Proof.* We show the inclusion $\subset$ first. Fix vectors $v \in N_Q(\bar{x})$ and $w \in T_Q(\bar{x})$; we aim to show $\langle v, w\rangle \leq 0$. By definition of the tangent cone, there exist sequences $x_i \to \bar{x}$ in $Q$ and $\tau_i \searrow 0$ satisfying $\tau_i^{-1}(x_i - \bar{x}) \to w$. We may suppose $w \neq 0$ (why?) and therefore $x_i \neq \bar{x}$ for all large $i$. The definition of the normal cones then yields

$$\langle v, w\rangle = \lim_{i\to\infty} \frac{\langle v, x_i - \bar{x}\rangle}{\tau_i} = \lim_{i\to\infty} \frac{\langle v, x_i - \bar{x}\rangle}{\|x_i - \bar{x}\|} \cdot \lim_{i\to\infty} \left\| \frac{x_i - \bar{x}}{\tau_i} \right\| \leq 0.$$

Since $w \in T_Q(\bar{x})$ was arbitrary, we deduce $v \in (T_Q(\bar{x}))^\circ$.

To see the reverse inclusion $\supset$, fix a vector $v \in (T_Q(\bar{x}))^\circ$. Thus the inequality $\langle v, w\rangle \leq 0$ holds for all $w \in T_Q(\bar{x})$. Consider now a sequence $x_i \to \bar{x}$ in $Q$, such that $x_i \neq \bar{x}$ for all large $i$. Defining $\tau_i = \|x_i - \bar{x}\|$, we deduce

$$\limsup_{i\to\infty} \frac{\langle v, x_i - \bar{x}\rangle}{\|x_i - \bar{x}\|} = \limsup_{i\to\infty} \ \langle v, \tau_i^{-1}(x_i - \bar{x})\rangle. \tag{2.6}$$

Passing to subsequence, we may assume that the real numbers $\langle v, \tau_i^{-1}(x_i - \bar{x})\rangle$ converge. Since the vectors $\tau_i^{-1}(x_i - \bar{x})$ all have unit norm, we may again pass to a subsequence to ensure that $\tau_i^{-1}(x_i - \bar{x})$ converge to some vector $w$. Since $w$ clearly lies in $T_Q(\bar{x})$ while $v$ lies in $(T_Q(\bar{x}))^\circ$, we deduce that the right-hand side of (2.6) is nonpositive. Therefore the inclusion $v \in N_Q(\bar{x})$ holds as claimed. $\square$

When $Q$ is convex, the definition of the normal cone simplifies.

**Exercise 2.35.** ✍ Show for any convex sets $Q \subset \mathbf{E}$ and a point $\bar{x} \in Q$ the equality

$$N_Q(\bar{x}) = \{v \in \mathbf{E} : \langle v, x - \bar{x}\rangle \leq 0 \quad \text{for all } x \in Q\}.$$

[**Hint:** Appeal to Lemma 2.34.]

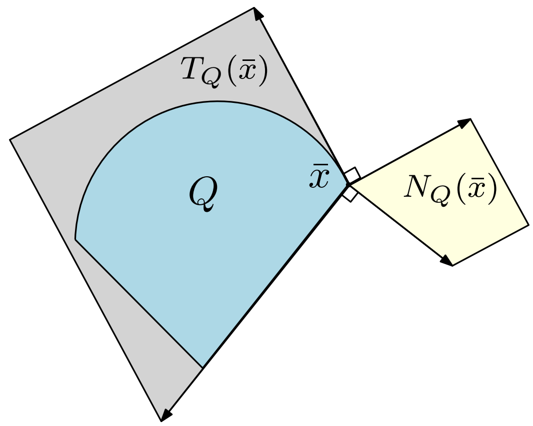



Figure 2.14: Illustration of the tangent and normal cones for a convex set.

Thus the $o(\|x-\bar{x}\|)$ error in the definition of the normal cone is irrelevant for convex set. That is, every vector $v \in N_Q(\bar{x})$ truly makes an obtuse angle with any direction $x - \bar{x}$ for $x \in Q$. Equivalently, every vector $v \in N_Q(\bar{x})$ corresponds to a halfspace $\{x : \langle v, x\rangle \leq \langle v, \bar{x}\rangle\}$ containing $Q$. See Figure 2.14.

Some thought shows that normal cones and nearest-point projections are intimately related. The following exercise explores this connection and will be used extensively in the sequel; see the companion Figure 2.15.

**Exercise 2.36.** ✍ Prove that the following properties are equivalent for any nonempty, closed, convex set $Q$, a point $\bar{x} \in Q$, and a vector $v \in \mathbf{E}$.

1. $v \in N_Q(\bar{x})$,
2. $\bar{x} \in \operatorname{argmax}_{x\in Q}\langle v, x\rangle$.
3. $\operatorname{proj}_Q(\bar{x} + \lambda v) = \bar{x}$ for all $\lambda \geq 0$,
4. $\operatorname{proj}_Q(\bar{x} + \bar{\lambda} v) = \bar{x}$ for some $\bar{\lambda} > 0$.

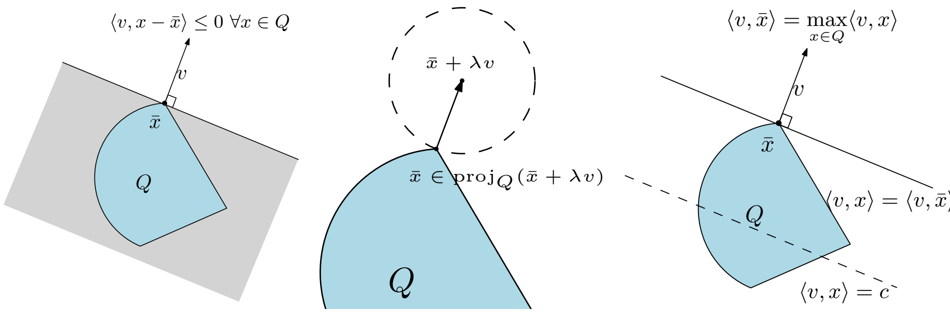
**Exercise 2.37.** Show for any convex cone $K$ and a point $x \in K$, the equality

$$N_K(x) = K^\circ \cap x^\perp.$$

**Exercise 2.38.** ✍ Show for any convex set $Q$ and a point $x \in Q$, the equivalence

$$x \in \operatorname{int} Q \quad \Longleftrightarrow \quad N_Q(x) = \{0\}.$$

What is the relationship between normal vectors $v \in N_Q(x)$ and supporting halfspaces, defined in Exercise 2.19?




(a) $v \in N_Q(\bar{x})$ (b) $\mathrm{proj}_Q(\bar{x} + \lambda v) = \bar{x}$ (c) $\bar{x} \in \mathrm{argmax}_{x\in Q}\langle v, x\rangle$

Figure 2.15: The figures depict the equivalences in Exercise 2.36.

## Comments

The results in Section 2.1-2.5 can be found in standard texts on convex geometry and analysis, such as Barvinok [2], Borwein-Lewis [3], Hiriart-Urruty and Lemaréchal [7], and Rockafellar [10]. The material in Section 2.6 blends the convex analytic viewpoint on tangency/normality with the more modern variation analytic perspective [12].

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