as $\alpha \to -\infty$, contradicting the nonnegativity of f. Our objective is to prove (b); that is, we want to show that for any $\mathbf{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$

$$\begin{pmatrix} \mathbf{y} \\ t \end{pmatrix}^T \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^T & c \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ t \end{pmatrix} \geq 0,$$

which is equivalent to

$$\mathbf{y}^T\mathbf{A}\mathbf{y} + 2t\mathbf{b}^T\mathbf{y} + ct^2 \geq 0. \tag{2.12}$$

To show the validity of (2.12) for any $\mathbf{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we consider two cases. If $t = 0$, then (2.12) reads as $\mathbf{y}^T\mathbf{A}\mathbf{y} \geq 0$, which is a valid inequality since we have shown that $\mathbf{A} \succeq 0$. The second case is when $t \neq 0$. To show that (2.12) holds in this case, note that (2.12) is the same as the inequality

$$t^2 f\left(\frac{\mathbf{y}}{t}\right) = t^2\left[\left(\frac{\mathbf{y}}{t}\right)^T \mathbf{A}\left(\frac{\mathbf{y}}{t}\right) + 2\mathbf{b}^T\left(\frac{\mathbf{y}}{t}\right) + c\right] \geq 0,$$

which holds true by the nonnegativity of f. □

Exercises

2.1. Find the global minimum and maximum points of the function $f(x,y) = x^2+y^2+2x-3y$ over the unit ball $S = B[0,1] = \{(x,y) : x^2+y^2 \leq 1\}$.

2.2. Let $\mathbf{a} \in \mathbb{R}^n$ be a nonzero vector. Show that the maximum of $\mathbf{a}^T\mathbf{x}$ over $B[0,1] = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$ is attained at $\mathbf{x}^* = \frac{\mathbf{a}}{\|\mathbf{a}\|}$ and that the maximal value is $\|\mathbf{a}\|$.

2.3. Find the global minimum and maximum points of the function $f(x,y) = 2x-3y$ over the set $S = \{(x,y) : 2x^2+5y^2 \leq 1\}$.

2.4. Show that if $\mathbf{A}, \mathbf{B}$ are $n \times n$ positive semidefinite matrices, then their sum $\mathbf{A}+\mathbf{B}$ is also positive semidefinite.

2.5. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{m\times m}$ be two symmetric matrices. Prove that the following two claims are equivalent:

(i) $\mathbf{A}$ and $\mathbf{B}$ are positive semidefinite.

(ii) $\begin{pmatrix} \mathbf{A} & \mathbf{0}_{n\times m} \\ \mathbf{0}_{m\times n} & \mathbf{B} \end{pmatrix}$ is positive semidefinite.

2.6. Let $\mathbf{B} \in \mathbb{R}^{n\times k}$ and let $\mathbf{A} = \mathbf{B}\mathbf{B}^T$.

(i) Prove $\mathbf{A}$ is positive semidefinite.

(ii) Prove that $\mathbf{A}$ is positive definite if and only if $\mathbf{B}$ has a full row rank.

2.7. (i) Let $\mathbf{A}$ be an $n \times n$ symmetric matrix. Show that $\mathbf{A}$ is positive semidefinite if and only if there exists a matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$ such that $\mathbf{A} = \mathbf{B}\mathbf{B}^T$.

(ii) Let $\mathbf{x} \in \mathbb{R}^n$ and let $\mathbf{A}$ be defined as

$$A_{ij} = x_i x_j, \quad i,j = 1,2,\ldots,n.$$

Show that $\mathbf{A}$ is positive semidefinite and that it is *not* a positive definite matrix when $n > 1$.

2.8. Let $\mathbf{Q} \in \mathbb{R}^{n\times n}$ be a positive definite matrix. Show that the "Q-norm" defined by

$$\|\mathbf{x}\|_{\mathbf{Q}} = \sqrt{\mathbf{x}^T\mathbf{Q}\mathbf{x}}$$

is indeed a norm.

2.9. Let $\mathbf{A}$ be an $n \times n$ positive semidefinite matrix.

(i) Show that for any $i \neq j$

$$A_{ii}A_{jj} \geq A_{ij}^2.$$

(ii) Show that if for some $i \in \{1, 2, \ldots, n\}$ $A_{ii} = 0$, then the ith row of $\mathbf{A}$ consists of zeros.

2.10. Let $\mathbf{A}^\alpha$ be the $n \times n$ matrix ($n > 1$) defined by

$$A_{ij}^\alpha = \begin{cases} \alpha, & i = j, \\ 1, & i \neq j. \end{cases}$$

Show that $\mathbf{A}^\alpha$ is positive semidefinite if and only if $\alpha \geq 1$.

2.11. Let $\mathbf{d} \in \Delta_n$ (Δ_n being the unit-simplex). Show that the $n \times n$ matrix $\mathbf{A}$ defined by

$$A_{ij} = \begin{cases} d_i - d_i^2, & i = j, \\ -d_i d_j, & i \neq j, \end{cases}$$

is positive semidefinite.

2.12. Prove that a 2×2 matrix $\mathbf{A}$ is negative semidefinite if and only if $\mathrm{Tr}(\mathbf{A}) \leq 0$ and $\det(\mathbf{A}) \geq 0$.

2.13. For each of the following matrices determine whether they are positive/negative semidefinite/definite or indefinite:

(i) $\mathbf{A} = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$.

(ii) $\mathbf{B} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix}$.

(iii) $\mathbf{C} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$.

(iv) $\mathbf{D} = \begin{pmatrix} -5 & 1 & 1 \\ 1 & -7 & 1 \\ 1 & 1 & -5 \end{pmatrix}$.

2.14. **(Schur complement lemma)** Let

$$\mathbf{D} = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^T & c \end{pmatrix},$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{b} \in \mathbb{R}^n, c \in \mathbb{R}$. Suppose that $\mathbf{A} \succ 0$. Prove that $\mathbf{D} \succeq 0$ if and only if $c - \mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} \geq 0$.

2.15. For each of the following functions, determine whether it is coercive or not:

(i) $f(x_1,x_2)=x_1^4+x_2^4$.
(ii) $f(x_1,x_2)=e^{x_1^2}+e^{x_2^2}-x_1^{200}-x_2^{200}$.
(iii) $f(x_1,x_2)=2x_1^2-8x_1x_2+x_2^2$.
(iv) $f(x_1,x_2)=4x_1^2+2x_1x_2+2x_2^2$.
(v) $f(x_1,x_2,x_3)=x_1^3+x_2^3+x_3^3$.
(vi) $f(x_1,x_2)=x_1^2-2x_1x_2^2+x_2^4$.
(vii) $f(\mathbf{x})=\frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\|\mathbf{x}\|+1}$, where $\mathbf{A}\in\mathbb{R}^{n\times n}$ is positive definite.

2.16. Find a function $f:\mathbb{R}^2\to\mathbb{R}$ which is not coercive and satisfies that for any $\alpha\in\mathbb{R}$

$$\lim_{|x_1|\to\infty} f(x_1,\alpha x_1)=\lim_{|x_2|\to\infty} f(\alpha x_2,x_2)=\infty.$$

2.17. For each of the following functions, find all the stationary points and classify them according to whether they are saddle points, strict/nonstrict local/global minimum/maximum points:

(i) $f(x_1,x_2)=(4x_1^2-x_2)^2$.
(ii) $f(x_1,x_2,x_3)=x_1^4-2x_1^2+x_2^2+2x_2x_3+2x_3^2$.
(iii) $f(x_1,x_2)=2x_2^3-6x_2^2+3x_1^2x_2$.
(iv) $f(x_1,x_2)=x_1^4+2x_1^2x_2+x_2^2-4x_1^2-8x_1-8x_2$.
(v) $f(x_1,x_2)=(x_1-2x_2)^4+64x_1x_2$.
(vi) $f(x_1,x_2)=2x_1^2+3x_2^2-2x_1x_2+2x_1-3x_2$.
(vii) $f(x_1,x_2)=x_1^2+4x_1x_2+x_2^2+x_1-x_2$.

2.18. Let f be twice continuously differentiable function over $\mathbb{R}^n$. Suppose that $\nabla^2 f(\mathbf{x})\succ 0$ for any $\mathbf{x}\in\mathbb{R}^n$. Prove that a stationary point of f is necessarily a strict global minimum point.

2.19. Let $f(\mathbf{x})=\mathbf{x}^T\mathbf{A}\mathbf{x}+2\mathbf{b}^T\mathbf{x}+c$, where $\mathbf{A}\in\mathbb{R}^{n\times n}$ is symmetric, $\mathbf{b}\in\mathbb{R}^n$, and $c\in\mathbb{R}$. Suppose that $\mathbf{A}\succeq 0$. Show that f is bounded below[1] over $\mathbb{R}^n$ if and only if $\mathbf{b}\in$ Range$(\mathbf{A})=\{\mathbf{A}\mathbf{y}:\mathbf{y}\in\mathbb{R}^n\}$.

[1]A function f is bounded below over a set C if there exists a constant α such that $f(\mathbf{x})\geq\alpha$ for any $\mathbf{x}\in C$.