Nonlinear Optimization **Homework 2 (Solutions)**
MATH 408 Spring 2019

1 Linear least squares problems

We will first focus on the linear least squares problem

$$\text{LLS} \qquad \min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax-b\|_2^2 \ ,$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

1. Listed below are two functions. In each case write the problem $\min_x f(x)$ as a linear least squares problem by specifying the matrix A and the vector b, and then solve the associated problem.

   (a) (2 points) $f(x) = (2x_1 - x_2 + 1)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$
   Most people ignored the $\frac{1}{2}$ in the objective function $f(x)$ above, yielding the following nicer expressions for A, b for the equivalent problem $\min_{x\in\mathbb{R}^n} \|Ax-b\|_2^2$.

   $$A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

   Solution: $\bar{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

   (b) (4 points) $f(x) = (1 - x_1)^2 + \sum_{j=1}^{3}(x_j - x_{j+1})^2$

   $$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or } A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

   Solution: $\bar{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

2. (5 points) Find the quadratic polynomial $p(t) = x_0 + x_1 t + x_2 t^2$ that best fits the following data in the least-squares sense:

$$\begin{array}{c|ccccc} t & -2 & -1 & 0 & 1 & 2 \\ \hline y & 2 & -10 & 0 & 2 & 1 \end{array}.$$

$A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ -10 \\ 0 \\ 2 \\ 1 \end{pmatrix}$. Solving the normal equations for the LLS problem specified by these yields $(x_0, x_1, x_2) = (-3, 1, 1)$ so the best fit quadratic is $p(t) = -3 + t + t^2$.

3. Consider the problem LLS with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) (2 point) What are the normal equations for this A and b.

$$A^T A x = A^T b$$

$$\begin{pmatrix} 4 & 0 & 4 \\ 0 & 4 & 4 \\ 4 & 4 & 8 \end{pmatrix} x = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

(b) (2 point) Solve the normal equations to obtain a solution to the problem LLS for this A and b.

Solution set to the normal equations is 1-dimensional and are given by $(3/4 - x_3, -1/4 - x_3, x_3)$ for $x_3 \in \mathbb{R}$

(c) (2 point) Write down the matrix that represents the orthogonal projection onto the range of A.

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(a) (2 point) Compute the orthogonal projection onto Ran(A).

$$P_{\text{Range}(A)} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

(b) (2 point) Compute the orthogonal projection onto Null(A^T).
By FTA, Null(A^T) = Range$(A)^\perp$ so

$$P_{\text{Null}(A^T)} = I_4 - \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

5. (5 points)[1] Generate thirty points (x_i, y_i) for $i = 1, \ldots, 30$, by the MATLAB code:

$$\begin{aligned} &\text{randn}('\text{seed}', 314); \\ &x = \text{linspace}(0, 1, 30); \\ &y = 2 \star x. \wedge 2 - 3 \star x + 1 + 0.05 \star \text{randn}(\text{size}(x)); \end{aligned}$$

Find the quadratic function $y = ax^2 + bx + c$ that best fits the points in the least squares sense. Indicate what are the parameters a,b,c found by the least squares solution, and plot the points along with the derived quadratic function.

Results will vary but you should find $a \approx 2, b \approx -3, c \approx 1$. You should have included your code and used the process of finding Vandermonde matrix and solving normal equations as discussed in class (not just using the built in polyfit command).

2 Quadratic optimization problems

Next, we will focus on the optimization problem

$$\mathcal{Q} \qquad \min_{x \in \mathbb{R}^n} \ \frac{1}{2} x^T H x + g^T x + b \ ,$$

where $H \in \mathbb{R}^{n \times n}$ is symmetric, $g \in \mathbb{R}^n$, and $b \in \mathbb{R}$.

1. Each of the following functions can be written in the form $f(x) = \frac{1}{2} x^T H x + g^T x + b$ with H symmetric. For each of these functions what are H and g.

[1] This is problem 3.2 in Beck's book.

(a) (2 points) $f(x) = x_1^2 - 4x_1 + 2x_2^2 + 7$

$$H = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad g = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \quad b = 7$$

(c) (2 points) $f(x) = x_1^2 - 2x_1x_2 + \frac{1}{2}x_2^2 - 8x_2$

$$H = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ -8 \end{pmatrix}, \quad b = 0$$

(d) (2 points) $f(x) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$

$$H = \begin{pmatrix} 8 & -4 & 0 \\ -4 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, \quad b = 1$$

(e) (2 points) $f(x) = x_1^2 + 16x_1x_2 + 4x_2x_3 + x_2^2$

$$H = \begin{pmatrix} 2 & 16 & 0 \\ 16 & 2 & 4 \\ 0 & 4 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad b = 0$$

2. Consider the matrix

$$H = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 9 & 3 \\ 2 & 3 & 4 \end{bmatrix}.$$

(a) (2 points) Compute the eigenvalues of H.

$$\lambda_1 = 12, \quad \lambda_2 = 3, \quad \lambda_3 = 2$$

(b) (2 points) Compute and orthonormal basis of eigenvectors for H.

$$u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

(c) (2 points) Compute the eigenvalue decomposition of H.

$$A = U \Lambda U^T \text{ where } U = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} \text{ from above and } \Lambda = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

3. For each of the matrices H and vectors g below determine the optimal value in $\mathcal{Q}$. If an optimal solution exists, compute the complete set of optimal solutions.

(a) (3 points)

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Solve $Hx = -g$ to find $\bar{x} = (-2, 1, -1)$ which yields optimal value -3.

(b) (3 points)

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

H is indefinite so optimal value is $-\infty$

(c) (3 points)

$$H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

Solution set: $\{(-1-s, 1+3s, s) : s \in \mathbb{R}\}$ with optimal value -1.

4. (5 points) Consider the matrix $H \in \mathbb{R}^{3\times 3}$ and vector $g \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Does there exists a vector $u \in \mathbb{R}^3$ such that $f(tu) \xrightarrow{t\uparrow\infty} -\infty$? If yes, construct u.

Let $u = \begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix}$ $in\,$Null(H) so that

$$\begin{aligned} f(tu) &= (tu)^T H(tu) + g^T(tu) \\ &= t^2 u^T H u + t g^T u \\ &= 0t^2 - 4t \end{aligned}$$

As $t \nearrow \infty$, $f(tu) \searrow -\infty$.

5. Determine whether the following matrices are positive definite, positive semi-definite, or neither. (2 points each)

$$(a)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \succ 0 \qquad (b)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\ indefinite$$

$$(c)\ H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \succeq 0 \qquad (d)\ H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \succeq 0\ .$$

6. (3 points)[2] Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $L \in \mathbb{R}^{p \times n}$, and $\lambda > 0$. Consider the regularized least squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|^2 + \lambda \|Lx\|^2.$$

Show that the problem has a unique solution if and only if $\mathrm{Null}(A) \cap \mathrm{Null}(L) = \{0\}$.

Answer: First rewrite the objective as

$$\|Ax - b\|^2 + \lambda \|Lx\|^2 = x^T(A^T A + \lambda L^T L)x - 2(A^T b)^T x + \|b\|^2$$

Recall that the above problem has a unique solution if and only if $A^T A + \lambda L^T L \succ 0$. Thus we wish to show that $A^T A + \lambda L^T L \succ 0$ if and only if $\mathrm{Null}(A) \cap \mathrm{Null}(L) = \{0\}$.

($\Longleftarrow$) Suppose $\mathrm{Null}(A) \cap \mathrm{Null}(L) = \{0\}$ and let $z \neq 0$. Then either $Az \neq 0$ or $Lz \neq 0$, hence either $\|Az\| > 0$ or $\|Lz\|$. In either case,

$$z^T(A^T A + \lambda L^T L)z = \|Az\|^2 + \lambda \|Lz\|^2 > 0$$

Therefore $A^T A + \lambda L^T L \succ 0$

($\Longrightarrow$) Suppose $A^T A + \lambda L^T L \succ 0$ and let $z \in \mathrm{Null}(A) \cap \mathrm{Null}(L)$. Observe that

$$z^T(A^T A + \lambda L^T L)z = \|Az\|^2 + \lambda \|Lz\|^2 = \|0\|^2 + \lambda \|0\|^2 = 0$$

which can only occur for $z = 0$, hence $\mathrm{Null}(A) \cap \mathrm{Null}(L) = \{0\}$

[2] This is problem 3.1 in Beck's book.