Nonlinear Optimization	Homework 2
MATH 408 Spring 2019	Due April 19, 2019

1 Linear least squares problems

We will first focus on the linear least squares problem

$$\text{LLS} \qquad \min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax-b\|_2^2 ,$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

1. Listed below are two functions. In each case write the problem $\min_x f(x)$ as a linear least squares problem by specifying the matrix A and the vector b, and then solve the associated problem.

   (a) (2 points) $f(x) = (2x_1 - x_2 + 1)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$

   (b) (4 points) $f(x) = (1 - x_1)^2 + \sum_{j=1}^{3}(x_j - x_{j+1})^2$

2. (5 points) Find the quadratic polynomial $p(t) = x_0 + x_1 t + x_2 t^2$ that best fits the following data in the least-squares sense:

$$\begin{array}{c|ccccc} t & -2 & -1 & 0 & 1 & 2 \\ \hline y & 2 & -10 & 0 & 2 & 1 \end{array} .$$

3. Consider the problem LLS with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} .$$

   (a) (2 point) What are the normal equations for this A and b.

   (b) (2 point) Solve the normal equations to obtain a solution to the problem LLS for this A and b.

   (c) (2 point) Write down the matrix that represents the orthogonal projection onto the range of A.

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} .$$

(a) (2 point) Compute the orthogonal projection onto $\text{Ran}(A)$.

(b) (2 point) Compute the orthogonal projection onto $\text{Null}(A^T)$.

5. (5 points)[1] Generate thirty points (x_i, y_i) for $i = 1, \dots, 30$, by the MATLAB code:

$$\begin{aligned}&\text{randn}('\text{seed}', 314);\\&x = \text{linspace}(0, 1, 30);\\&y = 2 \star x. \wedge 2 - 3 \star x + 1 + 0.05 \star \text{randn}(\text{size}(x));\end{aligned}$$

Find the quadratic function $y = ax^2 + bx + c$ that best fits the points in the least squares sense. Indicate what are the parameters a,b,c found by the least squares solution, and plot the points along with the derived quadratic function. The resulting plot should look like the one in Figure 1

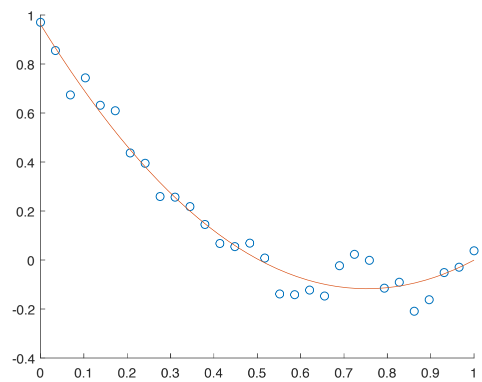

Figure 1: 30 points and their best quadratic least squares fit.

2 Quadratic optimization problems

Next, we will focus on the optimization problem

$$\mathcal{Q} \qquad \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T H x + g^T x + b \ ,$$

where $H \in \mathbb{R}^{n \times n}$ is symmetric, $g \in \mathbb{R}^n$, and $b \in \mathbb{R}$.

1. Each of the following functions can be written in the form $f(x) = \frac{1}{2} x^T H x + g^T x + b$ with H symmetric. For each of these functions what are H and g.

(a) (2 points) $f(x) = x_1^2 - 4x_1 + 2x_2^2 + 7$

[1]This is problem 3.2 in Beck's book.

(c) (2 points) $f(x) = x_1^2 - 2x_1x_2 + \frac{1}{2}x_2^2 - 8x_2$

(d) (2 points) $f(x) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$

(e) (2 points) $f(x) = x_1^2 + 16x_1x_2 + 4x_2x_3 + x_2^2$

2. Consider the matrix
$$H = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 9 & 3 \\ 2 & 3 & 4 \end{bmatrix}.$$

   (a) (2 points) Compute the eigenvalues of H.

   (b) (2 points) Compute and orthonormal basis of eigenvectors for H.

   (c) (2 points) Compute the eigenvalue decomposition of H.

3. For each of the matrices H and vectors g below determine the optimal value in $\mathcal{Q}$. If an optimal solution exists, compute the complete set of optimal solutions.

   (a) (3 points)
$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

   (b) (3 points)
$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

   (c) (3 points)
$$H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

4. Consider the matrix $H \in \mathbb{R}^{3\times 3}$ and vector $g \in \mathbb{R}^3$ given by
$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

   Does there exists a vector $u \in \mathbb{R}^3$ such that $f(tu) \xrightarrow{t\uparrow\infty} -\infty$? If yes, construct u.

5. Determine whether the following matrices are positive definite, positive semi-definite, or neither. (2 points each)

$$(a)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad (b)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$(c)\ H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \qquad (d)\ H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

6. (3 points)[2] Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $L \in \mathbb{R}^{p\times n}$, and $\lambda > 0$. Consider the regularized least squares problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|^2 + \lambda\|Lx\|^2.$$

Show that the problem has a unique solution if and only if $\mathrm{Null}(A) \cap \mathrm{Null}(L) = \{0\}$.

[2] This is problem 3.1 in Beck's book.