Nonlinear Optimization	Homework 1
MATH 408 Spring 2019	**Due April 12, 2019**

Exercises: Do the following exercises, justifying all steps.

1. Consider the system

$$\begin{array}{rcrcrcl} 4x_1 & & & - & x_3 & = & 200 \\ 9x_1 & + & x_2 & - & x_3 & = & 200 \\ 7x_1 & - & x_2 & + & 2x_3 & = & 200. \end{array}$$

   (a) (1 point) Write the augmented matrix corresponding to this system.

   (b) (3 point) Reduce the augmented system in part (a) to echelon form.

   (c) (2 point) Describe the set of solutions to the given system.

2. (6 points) Solve the following system of linear equations

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & & & = & 1 \\ -x_1 & - & 4x_2 & + & x_3 & = & 2 \\ & & 2x_2 & + & x_3 & = & 0. \end{array}$$

3. (3 points) Represent the linear span of the four vectors as the range space of some matrix:

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 5 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \quad x_4 = \begin{bmatrix} 7 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

4. (3 points) Compute a basis for $\mathrm{Nul}(A^T)^\perp$ where A is the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & 7 \\ 2 & 2 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 4 & 5 & -2 & 1 \end{bmatrix}.$$

5. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

   (a) (3 points) Find the eigenvalues of the matrix A. Is any eigenvalue repeated?

   (b) (4 points) Find three eigenvectors u_1, u_2, u_3 of A that are orthonormal.

   (c) (1 point) State a spectral (eigenvalue) decomposition of A.

6. Recall that for a function $f\colon \mathbb{R}^n \to \mathbb{R}$ the gradient is the vector of partial derivatives

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \frac{\partial f}{\partial x_2}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix}.$$

Compute the gradient of the following functions.

(a) (2 points) $f(x) = x_1^3 + x_2^3 - 3x_1 - 15x_2 + 25$ on $\mathbb{R}^2$

(b) (2 points) $f(x) = x_1^2 + x_2^2 - \sin(x_1 x_2)$ on $\mathbb{R}^2$

(c) (2 points) $f(x) = \|x\|^2 = \sum_{j=1}^n x_j^2$ on $\mathbb{R}^n$

(d) (2 points) $f(x) = e^{\|x\|^2}$ on $\mathbb{R}^n$

(e) (2 points) $f(x) = x_1 x_2 x_3 \cdots x_n$ on $\mathbb{R}^n$

(f) (2 points) $f(x) = -\log(x_1 x_2 x_3 \cdots x_n)$ on the set $\{x \in \mathbb{R}^n : x_i > 0 \text{ for all } i = 1, \ldots, n\}$

7. (12 points) Recall that that the Hessian matrix of a function $f\colon \mathbb{R}^d \to \mathbb{R}$ is the matrix of second partial derivatives:

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(x) \end{bmatrix}.$$

Compute the Hessian of the functions given in problem 6 above.