Nonlinear Optimization	Homework 5(Partial solutions)

Problem 8.2 Let $C = B[x_0, r]$, where $x_0 \in \mathbb{R}^n$ and $r > 0$ are given. Find a formula for the orthogonal projection operator P_C.

You can easily derive this formula from the projection onto $B[0, r]$ using an affine change of coordinates.

Problem 10.6 Consider the maximization problem

$$\begin{aligned} \max \quad & x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2 \\ s.t. \quad & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(1) Is the problem convex?
(2) Find all KKT points of the problem
(3) Find the optimal solution of the problem

Observe the maximization problem is equivalent to the minimization problem

$$\begin{aligned} -\min \quad & -\left(x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2\right) \\ s.t. \quad & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The Hessian of the objective $\begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$ is not psd, so the problem is not convex. The Lagrangian is

$$L(x_1, x_2, y_1, y_2, y_3) = -(x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1) + x_2 + y_1(x_1 + x_2 - 1) - y_2x_1 - y_3x_2$$

defined on $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2_+$. The KKT conditions are

$$\nabla_x L(x, y) = \begin{pmatrix} -2x_1 - 2x_2 + 3 + y_1 - y_2 \\ -2x_1 - 4x_2 - 1 + y_1 - y_3 \end{pmatrix} = 0 \tag{1}$$

$$y_2x_1 = 0 \tag{2}$$

$$y_3x_2 = 0 \tag{3}$$

$$x_1 + x_2 = 1 \tag{4}$$

$$x_1, x_2 \geq 0 \tag{5}$$

$$y_2, y_3 \geq 0 \tag{6}$$

Combining (4) with (1), we find $y_1 = y_2 - 1$ and $-2x_2 - 2 + y_1 - y_3 = 0$. If $y_2 = 0$, then $y_1 = -1$ and $-2x_2 - 3 - y_3 = 0$, but this cannot happen because $y_3 = -(2x_2 + 3)$ cannot be nonnegative if $x_2 \geq 0$. Thus to satisfy (2), we must have $x_1 = 0$, so $x_2 = 1$ by (4) and $y_3 = 0$ by (3). By (1), we get that $-2 + 3 + y_1 - y_2 = 0$ and $-4 - 1 + y_1 = 0$. Putting the pieces together, $(0,1)$ is the only KKT point with multipliers $(y_1, y_2, y_3) = (5, 6, 0)$. Certainly there exist feasible $\hat{x}$ such that $\hat{x}_1, \hat{x}_2 > 0$, so by Slater's condition, MFCQ holds for all feasible x and KKT are necessary conditions for optimality. Furthermore the extreme value theorem implies the existence of a global optimizer, so we conclude that the only KKT point $(0,1)$ solves the problem.

Problem 10.11 Use the KKT conditions to solve the problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ s.t. \quad & -2x_1 - x_2 + 10 \le 0 \\ & x_2 \ge 0 \end{aligned}$$

The objective is convex and the constraints are affine, hence the problem is convex. The Lagrangian is

$$L(x_1, x_2, y_1, y_2) = x_1^2 + x_2^2 + y_1(-2x_1 - x_2 + 10) - y_2 x_2$$

and the KKT conditions are

$$(1) \qquad \nabla_x L(x,y) = \begin{pmatrix} 2x_1 - 2y_1 \\ 2x_2 - y_1 - y_2 \end{pmatrix} = 0$$

$$(2) \qquad y_1(-2x_1 - x_2 + 10) = 0$$

$$(3) \qquad y_2 x_2 = 0$$

$$(4) \qquad -2x_1 - x_2 + 10 \le 0$$

$$(5) \qquad x_2 \ge 0$$

$$(6) \qquad y_1, y_2 \ge 0$$

If $y_1 = 0$, then $x_1 = 0$ and $y_2 = 2x_2$ by (1), which implies $x_2 = y_2 = 0$ by (3), but $(0,0)$ is not feasible.

If $y_1, y_2 \neq 0$, then $x_2 = 0$, and $-2x_1 - x_2 + 10 = 0$ implies $x_1 = 5$. Substituting into (1), the unique solution $x_1 = 5, x_2 = 0, y_1 = 5, y_2 = -5$ is dual infeasible

If $y_1 \neq 0$ and $y_2 = 0$, then the remaining system of linear equations has solution $x_1 = 4, x_2 = 2, y_1 = 4, y_2 = 0$, which is therefore a KKT point. Since this problem is convex, if you had found this point first, then sufficiency of the KKT conditions mean that you have found a globally optimal point and you can stop looking for more points.

Problem 11.1 Consider the optimization problem

$$\begin{aligned} \min \quad & x_1 - 4x_2 + x_3 \\ s.t. \quad & x_1 + 2x_2 + 2x_3 = -2 \\ & x_1^2 + x_2^2 + x_3^2 \le 1 \end{aligned}$$

The problem is convex so the KKT conditions are sufficient for optimality. There is a unique KKT point with irrational coordinates.

Problem 11.4 Consider the problem

$$\begin{aligned} \min \quad & x_1^2 - x_2^2 - x_3^2 \\ s.t. \quad & x_1^4 + x_2^4 + x_3^4 \leq 2 \end{aligned}$$

Problem is nonconvex and there are many KKT points. You must compare their respective values to determine optimal solution.