We can now use CVX to solve the equivalent problem (8.13):

```
cvx_begin
variable z(3)
minimize(d'*z-2*abs(f)'*sqrt(z))
subject to
sum(z)<=1
z>=0
cvx_end
```

The optimal solution is then computed by $y_i = -\text{sgn}(f_i)\sqrt{z_i}$ and then $\mathbf{x} = \mathbf{U}\mathbf{y}$:

```
>> y=-sign(f).*sqrt(z);
>> x=U*y
x =

   -0.2300
   -0.7259
    0.6482
```

Exercises

8.1. Consider the problem

$$(\text{P}) \quad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & g(\mathbf{x}) \leq 0 \\ & \mathbf{x} \in X, \end{array}$$

where f and g are convex functions over $\mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$ is a convex set. Suppose that $\mathbf{x}^*$ is an optimal solution of (P) that satisfies $g(\mathbf{x}^*) < 0$. Show that $\mathbf{x}^*$ is also an optimal solution of the problem

$$\begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X. \end{array}$$

8.2. Let $C = B[\mathbf{x}_0, r]$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and $r > 0$ are given. Find a formula for the orthogonal projection operator P_C.

8.3. Let f be a strictly convex function over $\mathbb{R}^m$ and let g be a convex function over $\mathbb{R}^n$. Define the function

$$h(\mathbf{x}) = f(\mathbf{A}\mathbf{x}) + g(\mathbf{x}),$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$. Assume that $\mathbf{x}^*$ and $\mathbf{y}^*$ are optimal solutions of the unconstrained problem of minimizing h. Show that $\mathbf{A}\mathbf{x}^* = \mathbf{A}\mathbf{y}^*$.

8.4. For each of the following optimization problems (a) show that it is convex, (b) write a CVX code that solves it, and (c) write down the optimal solution (by running CVX).

(i)

$$\begin{array}{ll} \min & x_1^2 + 2x_1x_2 + 2x_2^2 + x_3^2 + 3x_1 - 4x_2 \\ \text{s.t.} & \sqrt{2x_1^2 + x_1x_2 + 4x_2^2 + 4} + \frac{(x_1 - x_2 + x_3 + 1)^2}{x_1 + x_2} \leq 6 \\ & x_1, x_2, x_3 \geq 1. \end{array}$$

(ii)

$$\begin{array}{ll} \max & x_1 + x_2 + x_3 + x_4 \\ \text{s.t.} & (x_1 - x_2)^2 + (x_3 + 2x_4)^4 \le 5 \\ & x_1 + 2x_2 + 3x_3 + 4x_4 \le 6 \\ & x_1, x_2, x_3, x_4 \ge 0. \end{array}$$

(iii)

$$\begin{array}{ll} \min & 5x_1^2 + 4x_2^2 + 7x_3^2 + 4x_1x_2 + 2x_2x_3 + |x_1 - x_2| \\ \text{s.t.} & \frac{x_1^2+x_2^2}{x_3} + (x_1^2 + x_2^2 + 1)^4 \le 10 \\ & x_3 \ge 10. \end{array}$$

(iv)

$$\begin{array}{ll} \min & \sqrt{x_1^2 + x_2^2 + 2x_1 + 5} + x_1^2 + 2x_1x_2 + x_2^2 + 2x_1 + 3x_2 \\ \text{s.t.} & \frac{x_1^2}{x_1+x_2} + \left(\frac{x_1^2}{x_2} + 1\right)^8 \le 100 \\ & x_1 + x_2 \ge 4 \\ & x_2 \ge 1. \end{array}$$

(v)

$$\begin{array}{ll} \min & |2x_1 + 3x_2 + x_3| + x_1^2 + x_2^2 + x_3^2 + \sqrt{2x_1^2 + 4x_1x_2 + 7x_2^2 + 10x_2 + 6} \\ \text{s.t.} & \frac{x_1^2+1}{x_2} + 2x_1^2 + 5x_2^2 + 10x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3 \le 7 \\ & x_1 \ge 0 \\ & x_2 \ge 1. \end{array}$$

For this problem also show that the expression inside the square root is always nonnegative, i.e., $2x_1^2 + 4x_1x_2 + 7x_2^2 + 10x_2 + 6 \ge 0$ for all x_1, x_2.

(vi)

$$\begin{array}{ll} \min & \frac{1}{2x_2+3x_3} + 5x_1^2 + 4x_2^2 + 7x_3^2 + \frac{x_1^2+x_1+1}{x_2+x_3} \\ \text{s.t.} & \max\left\{x_1 + x_2, x_3^2\right\} + (x_1^2 + 4x_1x_2 + 5x_2^2 + 1)^2 \le 10 \\ & x_1, x_2, x_3 \ge 0.1. \end{array}$$

(vii)

$$\begin{array}{ll} \min & \sqrt{2x_1^2 + 3x_2^2 + x_3^2 + 4x_1x_2 + 7} + (x_1^2 + x_2^2 + x_3^2 + 1)^2 \\ \text{s.t.} & \frac{(x_1+x_2)^2}{x_3+1} + x_1^8 \le 7 \\ & x_1^2 + x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \le 10 \\ & x_1, x_2, x_3 \ge 0. \end{array}$$

(viii)

$$\begin{array}{ll} \min & \frac{x_1^4+2x_1^2x_2^2+x_2^4}{x_1^2+2x_1x_2+x_2^2} + \sqrt{x_3^2 + 1} \\ \text{s.t.} & x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \le 100 \\ & x_1 + x_2 + x_3 = 2 \\ & x_1 + x_2 \ge 1. \end{array}$$

(ix)

$$\begin{array}{ll} \min & \frac{x_1^4}{x_2^2} + \frac{x_2^4}{x_1^2} + 2x_1x_2 + |x_1 + 5| + |x_2 + 5| + |x_3 + 5| \\ \text{s.t.} & \left(\left(x_1^2 + x_2^2 + x_3^2 + 1\right)^2 + 1\right)^2 + x_1^4 + x_2^4 + x_3^4 \le 200 \\ & \max\left\{x_1^2 + 4x_1x_2 + 9x_2^2, x_1, x_2\right\} \le 40 \\ & x_1 \ge 1 \\ & x_2 \ge 1. \end{array}$$

(x)

$$\begin{array}{ll} \min & (x_1+x_2+x_3)^8+x_1^2+x_2^2+3x_3^2+2x_1x_2+2x_2x_3+2x_1x_3 \\ \text{s.t.} & (|x_1-2x_2|+1)^4+\frac{1}{x_3}\leq 10, \\ & 2x_1+2x_2+x_3\leq 1, \\ & 0\leq x_3\leq 1. \end{array}$$

8.5. Suppose that we are given 40 points in the plane. Each of these points belongs to one of two classes. Specifically, there are 19 points of class 1 and 21 points of class 2. The points are generated and plotted by the MATLAB commands

```
rand('seed',314);
x=rand(40,1);
y=rand(40,1);
class=[2*x<y+0.5]+1;
A1=[x(find(class==1)),y(find(class==1))];
A2=[x(find(class==2)),y(find(class==2))];
plot(A1(:,1),A1(:,2),'*','MarkerSize',6)
hold on
plot(A2(:,1),A2(:,2),'d','MarkerSize',6)
hold off
```

The plot of the points is given in Figure 8.8. Note that the rows of $\mathbf{A}_1 \in \mathbb{R}^{19\times 2}$ are the 19 points of class 1 and the rows of $\mathbf{A}_2 \in \mathbb{R}^{21\times 2}$ are the 21 points of class 2. Write a CVX-based code for finding the maximum-margin line separating the two classes of points.

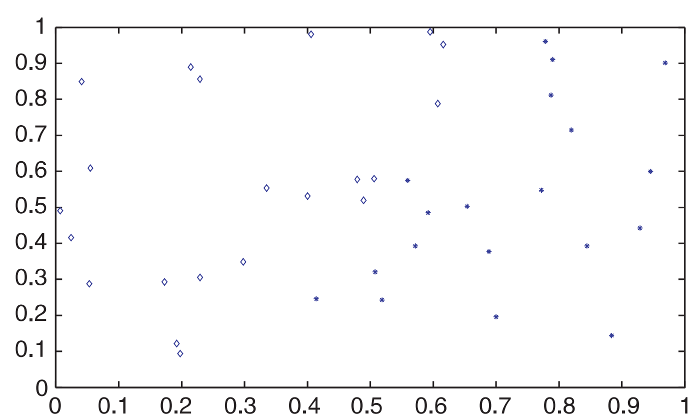

Figure 8.8. 40 *points of two classes: class* 1 *points are denoted by asterisks, and class* 2 *points are denoted by diamonds.*

Taking n to ∞ and using the continuity of f, we obtain that

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^*) = 0.$$

Now let $i \in I_0(\mathbf{x}^*)$. If there exist an infinite number of indices k_n for which $x_i^{k_n+1} \neq 0$, then as in the previous case, we obtain that $x_i^{k_n+1} = x_i^{k_n} - \frac{\partial f}{\partial x_i}(\mathbf{x}^{k_n})$ for these indices, implying (by taking the limit) that $\frac{\partial f}{\partial x_i}(\mathbf{x}^*) = 0$. In particular, $|\frac{\partial f}{\partial x_i}(\mathbf{x}^*)| \le LM_s(\mathbf{x}^*)$. On the other hand, if there exists an $M > 0$ such that for all $n > M$ $x_i^{k_n+1} = 0$, then

$$\left| x_i^{k_n} - \frac{1}{L}\frac{\partial f}{\partial x_i}(\mathbf{x}^{k_n}) \right| \le M_s\left(\mathbf{x}^{k_n} - \frac{1}{L}\nabla f(\mathbf{x}^{k_n})\right) = M_s(\mathbf{x}^{k_n+1}).$$

Thus, taking n to infinity, while exploiting the continuity of the function M_s, we obtain that

$$\left|\frac{\partial f}{\partial x_i}(\mathbf{x}^*)\right| \le LM_s(\mathbf{x}^*),$$

and hence, by Lemma 9.20, the desired result is established.

Exercises

9.1. Let f be a continuously differentiable convex function over a closed and convex set $C \subseteq \mathbb{R}^n$. Show that $\mathbf{x}^* \in C$ is an optimal solution of the problem

$$\text{(P)} \quad \min\{f(\mathbf{x}) : \mathbf{x} \in C\}$$

if and only if

$$\langle \nabla f(\mathbf{x}), \mathbf{x}^* - \mathbf{x} \rangle \le 0 \text{ for all } \mathbf{x} \in C.$$

9.2. Consider the Huber function

$$H_\mu(\mathbf{x}) = \begin{cases} \frac{\|\mathbf{x}\|^2}{2\mu}, & \|\mathbf{x}\| \le \mu, \\ \|\mathbf{x}\| - \frac{\mu}{2} & \text{else,} \end{cases}$$

where $\mu > 0$ is a given parameter. Show that $H_\mu \in C^{1,1}_{\frac{1}{\mu}}$.

9.3. Consider the minimization problem

$$\text{(Q)} \quad \begin{array}{ll} \min & 2x_1^2 + 3x_2^2 + 4x_3^2 + 2x_1x_2 - 2x_1x_3 - 8x_1 - 4x_2 - 2x_3 \\ \text{s.t.} & x_1, x_2, x_3 \ge 0. \end{array}$$

(i) Show that the vector $(\frac{17}{7}, 0, \frac{6}{7})^T$ is an optimal solution of (Q).

(ii) Employ the gradient projection method via MATLAB with constant stepsize $\frac{1}{L}$ (L being the Lipschitz constant of the gradient of the objective function). Show the function values of the first 100 iterations and the produced solution.

9.4. Consider the minimization problem

$$\text{(P)} \quad \min\{f(\mathbf{x}) : \mathbf{a}^T\mathbf{x} = 1, \mathbf{x} \in \mathbb{R}^n\},$$

where f is a continuously differentiable function over $\mathbb{R}^n$ and $\mathbf{a} \in \mathbb{R}^n_{++}$. Show that $\mathbf{x}^*$ satisfying $\mathbf{a}^T\mathbf{x}^* = 1$ is a stationary point of (P) if and only if

$$\frac{\frac{\partial f}{\partial x_1}(\mathbf{x}^*)}{a_1} = \frac{\frac{\partial f}{\partial x_2}(\mathbf{x}^*)}{a_2} = \cdots = \frac{\frac{\partial f}{\partial x_n}(\mathbf{x}^*)}{a_n}.$$

9.5. Consider the minimization problem

$$\text{(P)} \quad \min\{f(\mathbf{x}) : \mathbf{x} \in \Delta_n\},$$

where f is a continuously differentiable function over Δ_n. Prove that $\mathbf{x}^* \in \Delta_n$ is a stationary point of (P) if and only if there exists $\mu \in \mathbb{R}$ such that

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^*) \begin{cases} = \mu, & x_i^* > 0, \\ \geq \mu, & x_i^* = 0. \end{cases}$$

9.6. Let $S \subseteq \mathbb{R}^n$ be a nonempty closed and convex set, and let $f \in C_L^{1,1}(S)$ be a convex function over S. Assume that the optimal value of the problem

$$\text{(P)} \quad \min\{f(\mathbf{x}) : \mathbf{x} \in S\},$$

denoted by f^* is real. Prove that for any $\mathbf{x} \in S$ the following inequality holds:

$$f(\mathbf{x}) - f^* \geq \frac{1}{2L}\|G_L(\mathbf{x})\|^2,$$

where $G_L(\mathbf{x}) = L[\mathbf{x} - P_S(\mathbf{x} - \frac{1}{L}\nabla f(\mathbf{x}))]$.

9.7. Let $f \in C_L^{1,1}(S)$ be a strongly convex function over a nonempty closed convex set $S \subseteq \mathbb{R}^n$ with strong convexity parameter $\sigma > 0$, that is,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\sigma}{2}\|\mathbf{x} - \mathbf{y}\|^2 \text{ for all } \mathbf{x}, \mathbf{y} \in S.$$

Consider the problem

$$\text{(P)} \quad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in S, \end{array}$$

where S is a closed and convex set. Let $\{\mathbf{x}_k\}_{k \geq 0}$ be the sequence generated by the gradient projection method for solving problem (P) with a constant stepsize $t_k = \frac{1}{L}$. Let $\mathbf{x}^*$ be the optimal solution of (P), and let f^* be the optimal value of (P). Prove that there exists a constant $c \in (0, 1)$ such that for any $k \geq 0$

$$\|\mathbf{x}_{k+1} - \mathbf{x}^*\| \leq c\|\mathbf{x}_k - \mathbf{x}^*\|.$$

Find an explicit expression for c.

Therefore, by Lemma 1.12 an optimal solution of the problem is an eigenvector of the matrix $\mathbf{A}^T(\mathbf{I}_m - \frac{1}{m}\mathbf{e}\mathbf{e}^T)\mathbf{A}$ corresponding to the minimum eigenvalue; the optimal function value is the minimum eigenvalue $\lambda_{\min}[\mathbf{A}^T(\mathbf{I}_m - \frac{1}{m}\mathbf{e}\mathbf{e}^T)\mathbf{A}]$.

Exercises

10.1. Show that the dual cone of

$$M = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \geq \mathbf{0}\} \quad (\mathbf{A} \in \mathbb{R}^{m\times n})$$

is

$$M^* = \{\mathbf{A}^T\mathbf{v} : \mathbf{v} \in \mathbb{R}^m_+\}.$$

10.2. **(nonhomogenous Farkas' lemma)** Let $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{c} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m$, and $d \in \mathbb{R}$. Suppose that there exists $\mathbf{y} \geq \mathbf{0}$ such that $\mathbf{A}^T\mathbf{y} = \mathbf{c}$. Prove that exactly one of the following two systems is feasible:

A. $\mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{c}^T\mathbf{x} > d$.

B. $\mathbf{A}^T\mathbf{y} = \mathbf{c}, \mathbf{b}^T\mathbf{y} \leq d, \mathbf{y} \geq \mathbf{0}$.

10.3. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{c} \in \mathbb{R}^n$. Show that exactly one of the following two systems is feasible:

A. $\mathbf{A}\mathbf{x} \geq \mathbf{0}, \mathbf{x} \geq \mathbf{0}, \mathbf{c}^T\mathbf{x} > 0$.

B. $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}, \mathbf{y} \leq \mathbf{0}$.

10.4. Prove Motzkin's theorem of the alternative: the system

$$\text{(I)} \quad \begin{array}{lcl} \mathbf{A}\mathbf{d} & < & \mathbf{0}, \\ \mathbf{B}\mathbf{d} & \leq & \mathbf{0} \end{array}$$

has a solution if and only if the system

$$\text{(II)} \quad \begin{array}{l} \mathbf{A}^T\mathbf{u} + \mathbf{B}^T\mathbf{y} = \mathbf{0}, \\ \mathbf{u}, \mathbf{y} \geq \mathbf{0}, \qquad \mathbf{u} \neq \mathbf{0} \end{array}$$

does not have a solution (here $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{B} \in \mathbb{R}^{k\times n}$).

10.5. Prove the following nonhomogenous version of Gordan's alternative theorem: Given $\mathbf{A} \in \mathbb{R}^{m\times n}$, exactly one of these two systems is feasible.

A. $\mathbf{A}\mathbf{z} < \mathbf{b}$.

B. $\mathbf{A}^T\mathbf{y} = \mathbf{0}, \mathbf{b}^T\mathbf{y} \leq 0, \mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{0}$.

10.6. Consider the maximization problem

$$\begin{array}{ll} \max & x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2 \\ \text{s.t.} & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0. \end{array}$$

(i) Is the problem convex?

(ii) Find all the KKT points of the problem.

(iii) Find the optimal solution of the problem.

10.7. Consider the problem

$$\begin{aligned} \min \quad & -x_1 x_2 x_3 \\ \text{s.t.} \quad & x_1 + 3x_2 + 6x_3 \leq 48, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(i) Write the KKT conditions for the problem.

(ii) Find the optimal solution of the problem.

10.8. Consider the problem

$$\begin{aligned} \min \quad & x_1^2 + 2x_2^2 + x_1 \\ \text{s.t.} \quad & x_1 + x_2 \leq a, \end{aligned}$$

where $a \in \mathbb{R}$ is a parameter.

(i) Prove that for any $a \in \mathbb{R}$, the problem has a unique optimal solution (without actually solving it).

(ii) Solve the problem (the solution will be in terms of the parameter a).

(iii) Let $f(a)$ be the optimal value of the problem with parameter a. Write an explicit expression for f and prove that it is a convex function.

10.9. Consider the problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 - 2x_1 - 4x_2 - 6x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \leq 1. \end{aligned}$$

(i) Is the problem convex?

(ii) Find all the KKT points of the problem.

(iii) Find the optimal solution of the problem.

10.10. Consider the problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 + x_3^2 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \geq 4 \\ & x_3 \leq 1. \end{aligned}$$

(i) Write down the KKT conditions.

(ii) Without solving the KKT system, prove that the problem has a unique optimal solution and that this solution satisfies the KKT conditions.

(iii) Find the optimal solution of the problem using the KKT system.

10.11. Use the KKT conditions in order to solve the problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & -2x_1 - x_2 + 10 \leq 0 \\ & x_2 \geq 0. \end{aligned}$$

Exercises

11.1. Consider the optimization problem

$$\text{(P)} \quad \begin{array}{ll} \min & x_1 - 4x_2 + x_3 \\ \text{s.t.} & x_1 + 2x_2 + 2x_3 = -2, \\ & x_1^2 + x_2^2 + x_3^2 \le 1. \end{array}$$

(i) Given a KKT point of problem (P), must it be an optimal solution?

(ii) Find the optimal solution of the problem using the KKT conditions.

11.2. Consider the optimization problem

$$\text{(P)} \quad \min\{\mathbf{a}^T\mathbf{x} : \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c \le 0\},$$

where $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is positive definite, $\mathbf{a}(\neq \mathbf{0}), \mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

(i) For which values of $\mathbf{Q}, \mathbf{b}, \mathbf{c}$ is the problem feasible?

(ii) For which values of $\mathbf{Q}, \mathbf{b}, \mathbf{c}$ are the KKT conditions necessary?

(iii) For which values of $\mathbf{Q}, \mathbf{b}, \mathbf{c}$ are the KKT conditions sufficient?

(iv) Under the condition of part (ii), find the optimal solution of (P) using the KKT conditions.

11.3. Consider the optimization problem

$$\begin{array}{ll} \min & x_1^4 - 2x_2^2 - x_2 \\ \text{s.t.} & x_1^2 + x_2^2 + x_2 \le 0. \end{array}$$

(i) Is the problem convex?

(ii) Prove that there exists an optimal solution to the problem.

(iii) Find all the KKT points. For each of the points, determine whether it satisfies the second order necessary conditions.

(iv) Find the optimal solution of the problem.

11.4. Consider the optimization problem

$$\begin{array}{ll} \min & x_1^2 - x_2^2 - x_3^2 \\ \text{s.t.} & x_1^4 + x_2^4 + x_3^4 \le 1. \end{array}$$

(i) Is the problem convex?

(ii) Find all the KKT points of the problem.

(iii) Find the optimal solution of the problem.

11.5. Consider the optimization problem

$$\begin{array}{ll} \min & -2x_1^2 + 2x_2^2 + 4x_1 \\ \text{s.t.} & x_1^2 + x_2^2 - 4 \le 0, \\ & x_1^2 + x_2^2 - 4x_1 + 3 \le 0. \end{array}$$

(i) Prove that there exists an optimal solution to the problem.

(ii) Find all the KKT points.

(iii) Find the optimal solution of the problem.

11.6. Use the KKT conditions in order to find an optimal solution of the each of the following problems:

(i)

$$\begin{array}{ll} \min & 3x_1^2 + x_2^2 \\ \text{s.t.} & x_1 - x_2 + 8, \le 0, \\ & x_2 \ge 0. \end{array}$$

(ii)

$$\begin{array}{ll} \min & 3x_1^2 + x_2^2 \\ \text{s.t.} & 3x_1^2 + x_2^2 + x_1 + x_2 + 0.1 \le 0, \\ & x_2 + 10 \ge 0. \end{array}$$

(iii)

$$\begin{array}{ll} \min & 2x_1 + x_2 \\ \text{s.t.} & 4x_1^2 + x_2^2 - 2 \le 0, \\ & 4x_1 + x_2 + 3 \le 0. \end{array}$$

(iv)

$$\begin{array}{ll} \min & x_1^3 + x_2^3 \\ \text{s.t.} & x_1^2 + x_2^2 \le 1. \end{array}$$

(v)

$$\begin{array}{ll} \min & x_1^4 - x_2^2 \\ \text{s.t.} & x_1^2 + x_2^2 \le 1, \\ & 2x_2 + 1 \le 0. \end{array}$$

11.7. Let $a > 0$. Find all the optimal solutions of

$$\max\{x_1 x_2 x_3 : a^2 x_1^2 + x_2^2 + x_3^2 \le 1\}.$$

11.8. (i) Find a formula for the orthogonal projection of a vector $\mathbf{y} \in \mathbb{R}^3$ onto the set

$$C = \{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + 2x_2^2 + 3x_3^2 \le 1\}.$$

The formula should depend on a single parameter that is a root of a strictly decreasing one-dimensional function.

(ii) Write a MATLAB function whose input is a three-dimensional vector and its output is the orthogonal projection of the input onto C.

11.9. Consider the optimization problem

$$\text{(P)} \quad \begin{array}{ll} \min & 2x_1 x_2 + \frac{1}{2}x_3^2 \\ \text{s.t.} & 2x_1 x_3 + \frac{1}{2}x_2^2 \le 0, \\ & 2x_2 x_3 + \frac{1}{2}x_1^2 \le 0. \end{array}$$

(i) Show that the optimal solution of problem (P) is $\mathbf{x}^* = (0,0,0)$.

(ii) Show that $\mathbf{x}^*$ does not satisfy the second order necessary optimality conditions.

11.10. Consider the convex optimization problem

$$\text{(P)} \quad \begin{array}{ll} \min & f_0(\mathbf{x}) \\ \text{s.t.} & f_i(\mathbf{x}) \leq 0, \quad i = 1,2,\ldots,m, \end{array}$$

where f_0 is a continuously differentiable convex function and $f_1, f_2, \ldots, f_m$ are continuously differentiable *strictly* convex functions. Let $\mathbf{x}^*$ be a feasible solution of (P). Suppose that the following condition is satisfied: there exist $y_i \geq 0, i \in \{0\} \cup I(\mathbf{x}^*)$, which are not all zeros such that

$$y_0 \nabla f_0(\mathbf{x}^*) + \sum_{i \in I(\mathbf{x}^*)} y_i \nabla f_i(\mathbf{x}^*) = 0.$$

Prove that $\mathbf{x}^*$ is an optimal solution of (P).

11.11. Consider the optimization problem

$$\begin{array}{ll} \min & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & f_i(\mathbf{x}) \leq 0, \quad i = 1,2,\ldots,m, \end{array}$$

where $\mathbf{c} \neq 0$ and $f_1, f_2, \ldots, f_m$ are continuous over $\mathbb{R}^n$. Prove that if $\mathbf{x}^*$ is a local minimum of the problem, then $I(\mathbf{x}^*) \neq \emptyset$.

11.12. Consider the QCQP problem

$$\text{(QCQP)} \quad \begin{array}{ll} \min & \mathbf{x}^T \mathbf{A}_0 \mathbf{x} + 2\mathbf{b}_0^T \mathbf{x} \\ \text{s.t.} & \mathbf{x}^T \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^T \mathbf{x} + c_i \leq 0, \quad i = 1,2,\ldots,m, \end{array}$$

where $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathbb{R}^{n \times n}$ are symmetric matrices, $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m \in \mathbb{R}^n$, and $c_1, c_2, \ldots, c_m \in \mathbb{R}$. Suppose that $\mathbf{x}^*$ satisfies the following condition: there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ such that

$$\begin{aligned} \left(\mathbf{A}_0 + \sum_{i=1}^m \lambda_i \mathbf{A}_i\right)\mathbf{x}^* + \left(\mathbf{b}_0 + \sum_{i=1}^m \lambda_i \mathbf{b}_i\right) &= 0, \\ \lambda_i \left[(\mathbf{x}^*)^T \mathbf{A}_i (\mathbf{x}^*) + 2\mathbf{b}_i^T \mathbf{x}^* + c_i\right] &= 0, \quad i = 1,2,\ldots,m, \\ (\mathbf{x}^*)^T \mathbf{A}_i (\mathbf{x}^*) + 2\mathbf{b}_i^T \mathbf{x}^* + c_i &\leq 0, \quad i = 1,2,\ldots,m, \\ \mathbf{A}_0 + \sum_{i=1}^m \lambda_i \mathbf{A}_i &\succeq 0. \end{aligned}$$

Prove that $\mathbf{x}^*$ is an optimal solution of (QCQP).