Finally, using the latter inequality along with the fact that for every $k = 0, 1, \ldots, n$ the obvious inequality $\|\nabla f(\mathbf{x}_k)\|^2 \geq \min_{k=0,1,\ldots,n} \|\nabla f(\mathbf{x}_k)\|^2$ holds, it follows that

$$f(\mathbf{x}_0) - f^* \geq M(n+1) \min_{k=0,1,\ldots,n} \|\nabla f(\mathbf{x}_k)\|^2,$$

implying the desired result. □

Exercises

4.1. Let $f \in C_L^{1,1}(\mathbb{R}^n)$ and let $\{\mathbf{x}_k\}_{k\geq 0}$ be the sequence generated by the gradient method with a constant stepsize $t_k = \frac{1}{L}$. Assume that $\mathbf{x}_k \to \mathbf{x}^*$. Show that if $\nabla f(\mathbf{x}_k) \neq 0$ for all $k \geq 0$, then $\mathbf{x}^*$ is *not* a local maximum point.

4.2. [9, Exercise 1.3.3] Consider the minimization problem

$$\min\{\mathbf{x}^T \mathbf{Q} \mathbf{x} : \mathbf{x} \in \mathbb{R}^2\},$$

where $\mathbf{Q}$ is a positive definite 2×2 matrix. Suppose we use the diagonal scaling matrix

$$\mathbf{D} = \begin{pmatrix} Q_{11}^{-1} & 0 \\ 0 & Q_{22}^{-1} \end{pmatrix}.$$

Show that the above scaling matrix improves the condition number of $\mathbf{Q}$ in the sense that

$$\varkappa(\mathbf{D}^{1/2}\mathbf{Q}\mathbf{D}^{1/2}) \leq \varkappa(\mathbf{Q}).$$

4.3. Consider the quadratic minimization problem

$$\min\{\mathbf{x}^T \mathbf{A} \mathbf{x} : \mathbf{x} \in \mathbb{R}^5\},$$

where $\mathbf{A}$ is the 5×5 Hilbert matrix defined by

$$A_{i,j} = \frac{1}{i+j-1}, \quad i, j = 1, 2, 3, 4, 5.$$

The matrix can be constructed via the MATLAB command `A = hilb(5)`. Run the following methods and compare the number of iterations required by each of the methods when the initial vector is $\mathbf{x}_0 = (1, 2, 3, 4, 5)^T$ to obtain a solution $\mathbf{x}$ with $\|\nabla f(\mathbf{x})\| \leq 10^{-4}$:

- gradient method with backtracking stepsize rule and parameters $\alpha = 0.5, \beta = 0.5, s = 1$;
- gradient method with backtracking stepsize rule and parameters $\alpha = 0.1, \beta = 0.5, s = 1$;
- gradient method with exact line search;
- diagonally scaled gradient method with diagonal elements $D_{ii} = \frac{1}{A_{ii}}, i = 1, 2, 3, 4, 5$ and exact line search;
- diagonally scaled gradient method with diagonal elements $D_{ii} = \frac{1}{A_{ii}}, i = 1, 2, 3, 4, 5$ and backtracking line search with parameters $\alpha = 0.1, \beta = 0.5, s = 1$.

Downloaded 04/29/19 to 128.208.201.54. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

(i) Show that as long as all the points $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ do not reside on the same line in the plane, the method is well-defined, meaning that the linear least squares problem solved at each iteration has a unique solution.

(ii) Write a MATLAB function that implements the damped Gauss–Newton method employed on problem (SL2) with a backtracking line search strategy with parameters $s = 1, \alpha = \beta = 0.5, \varepsilon = 10^{-4}$. Run the function on the two-dimensional problem ($n = 2$) with 5 anchors ($m = 5$) and data generated by the MATLAB commands

```
randn('seed',317);
A=randn(2,5);
x=randn(2,1);
d=sqrt(sum((A-x*ones(1,5)).^2))+0.05*randn(1,5);
d=d';
```

The columns of the 2×5 matrix $\mathbf{A}$ are the locations of the five sensors, $\mathbf{x}$ is the "true" location of the source, and $\mathbf{d}$ is the vector of noisy measurements between the source and the sensors. Compare your results (e.g., number of iterations) to the gradient method with backtracking and parameters $s = 1, \alpha = \beta = 0.5, \varepsilon = 10^{-4}$. Start both methods with the initial vector $(1000, -500)^T$.

4.7. Let $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$, where $\mathbf{A}$ is a symmetric $n \times n$ matrix, $\mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Show that the *smallest* Lipschitz constant of ∇f is $2\|\mathbf{A}\|$.

4.8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(\mathbf{x}) = \sqrt{1 + \|\mathbf{x}\|^2}$. Show that $f \in C_1^{1,1}$.

4.9. Let $f \in C_L^{1,1}(\mathbb{R}^m)$, and let $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$. Show that the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(\mathbf{x}) = f(\mathbf{A}\mathbf{x} + \mathbf{b})$ satisfies $g \in C_{\tilde{L}}^{1,1}(\mathbb{R}^n)$, where $\tilde{L} = \|\mathbf{A}\|^2 L$.

4.10. Give an example of a function $f \in C_L^{1,1}(\mathbb{R})$ and a starting point $x_0 \in \mathbb{R}$ such that the problem $\min f(x)$ has an optimal solution and the gradient method with constant stepsize $t = \frac{2}{L}$ diverges.

4.11. Suppose that $f \in C_L^{1,1}(\mathbb{R}^n)$ and assume that $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for any $\mathbf{x} \in \mathbb{R}^n$. Suppose that the optimal value of the problem $\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$ is f^*. Let $\{\mathbf{x}_k\}_{k \geq 0}$ be the sequence generated by the gradient method with constant stepsize $\frac{1}{L}$. Show that if $\{\mathbf{x}_k\}_{k \geq 0}$ is bounded, then $f(\mathbf{x}_k) \to f^*$ as $k \to \infty$.

Downloaded 04/29/19 to 128.208.201.54. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

5.2. Consider the Freudenstein and Roth test function

$$f(\mathbf{x}) = f_1(\mathbf{x})^2 + f_2(\mathbf{x})^2, \qquad \mathbf{x} \in \mathbb{R}^2,$$

where

$$f_1(\mathbf{x}) = -13 + x_1 + ((5 - x_2)x_2 - 2)x_2,$$
$$f_2(\mathbf{x}) = -29 + x_1 + ((x_2 + 1)x_2 - 14)x_2.$$

(i) Show that the function f has three stationary points. Find them and prove that one is a global minimizer, one is a strict local minimum and the third is a saddle point.

(ii) Use MATLAB to employ the following three methods on the problem of minimizing f:

1. the gradient method with backtracking and parameters $(s, \alpha, \beta) = (1, 0.5, 0.5)$.
2. the hybrid Gradient-Newton Method with parameters $(s, \alpha, \beta) = (0.5, 0.5)$.
3. damped Gauss–Newton's method with a backtracking line search strategy with parameters $(s, \alpha, \beta) = (1, 0.5, 0.5)$.

All the algorithms should use the stopping criteria $\|\nabla f(\mathbf{x})\| \leq 10^{-5}$. Each algorithm should be employed four times on the following four starting points: $(-50, 7)^T, (20, 7)^T, (20, -18)^T, (5, -10)^T$. For each of the four starting points, compare the number of iterations and the point to which each method converged. If a method did not converge, explain why.

5.3. Let f be a twice continuously differentiable function satisfying $L\mathbf{I} \succeq \nabla^2 f(\mathbf{x}) \succeq m\mathbf{I}$ for some $L > m > 0$ and let $\mathbf{x}^*$ be the unique minimizer of f over $\mathbb{R}^n$.

(i) Show that

$$f(\mathbf{x}) - f(\mathbf{x}^*) \geq \frac{m}{2}\|\mathbf{x} - \mathbf{x}^*\|^2$$

for any $\mathbf{x} \in \mathbb{R}^n$.

(ii) Let $\{\mathbf{x}_k\}_{k \geq 0}$ be the sequence generated by damped Newton's method with constant stepsize $t_k = \frac{m}{L}$. Show that

$$f(\mathbf{x}_k) - f(\mathbf{x}_{k+1}) \geq \frac{m}{2L} \nabla f(\mathbf{x}_k)^T (\nabla^2 f(\mathbf{x}_k))^{-1} \nabla f(\mathbf{x}_k).$$

(iii) Show that $\mathbf{x}_k \to \mathbf{x}^*$ as $k \to \infty$.

Downloaded 04/29/19 to 128.208.201.54. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Proof. For $p = 1$, the inequality follows by summing up the inequalities $|x_i + y_i| \leq |x_i| + |y_i|$. Suppose then that $p > 1$. We can assume that $\mathbf{x} \neq \mathbf{0}, \mathbf{y} \neq \mathbf{0}$, and $\mathbf{x}+\mathbf{y} \neq \mathbf{0}$. Otherwise, the inequality is trivial. The function $\varphi(t) = t^p$ is convex over $\mathbb{R}_+$ since $\varphi''(t) = p(p-1)t^{p-2} > 0$ for $t > 0$. Therefore, by the definition of convexity we have that for any $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ one has

$$(\lambda_1 t + \lambda_2 s)^p \leq \lambda_1 t^p + \lambda_2 s^p.$$

Let $i \in \{1,2,\ldots,n\}$. Plugging $\lambda_1 = \frac{||\mathbf{x}||_p}{||\mathbf{x}||_p+||\mathbf{y}||_p}, \lambda_2 = \frac{||\mathbf{y}||_p}{||\mathbf{x}||_p+||\mathbf{y}||_p}, t = \frac{|x_i|}{||\mathbf{x}||_p}$, and $s = \frac{|y_i|}{||\mathbf{y}||_p}$ in the above inequality yields

$$\frac{1}{(||\mathbf{x}||_p + ||\mathbf{y}||_p)^p}(|x_i| + |y_i|)^p \leq \frac{||\mathbf{x}||_p}{||\mathbf{x}||_p + ||\mathbf{y}||_p}\frac{|x_i|^p}{||\mathbf{x}||_p^p} + \frac{||\mathbf{y}||_p}{||\mathbf{x}||_p + ||\mathbf{y}||_p}\frac{|y_i|^p}{||\mathbf{y}||_p^p}.$$

Summing the above inequality over $i = 1,2,\ldots,n$, we obtain that

$$\frac{1}{(||\mathbf{x}||_p + ||\mathbf{y}||_p)^p}\sum_{i=1}^n(|x_i| + |y_i|)^p \leq \frac{||\mathbf{x}||_p}{||\mathbf{x}||_p + ||\mathbf{y}||_p} + \frac{||\mathbf{y}||_p}{||\mathbf{x}||_p + ||\mathbf{y}||_p} = 1,$$

and hence

$$\sum_{i=1}^n(|x_i| + |y_i|)^p \leq (||\mathbf{x}||_p + ||\mathbf{y}||_p)^p.$$

Finally,

$$||\mathbf{x}+\mathbf{y}||_p = \sqrt[p]{\sum_{i=1}^n |x_i + y_i|^p} \leq \sqrt[p]{\sum_{i=1}^n(|x_i| + |y_i|)^p} \leq ||\mathbf{x}||_p + ||\mathbf{y}||_p. \quad \square$$

Exercises

7.1. For each of the following sets determine whether they are convex or not (explaining your choice).

(i) $C_1 = \left\{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}||^2 = 1\right\}$.

(ii) $C_2 = \left\{\mathbf{x} \in \mathbb{R}^n : \max_{i=1,2,\ldots,n} x_i \leq 1\right\}$.

(iii) $C_3 = \left\{\mathbf{x} \in \mathbb{R}^n : \min_{i=1,2,\ldots,n} x_i \leq 1\right\}$.

(iv) $C_4 = \left\{\mathbf{x} \in \mathbb{R}^n_{++} : \prod_{i=1}^n x_i \geq 1\right\}$.

7.2. Show that the set

$$M = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{Q}\mathbf{x} \leq (\mathbf{a}^T\mathbf{x})^2, \mathbf{a}^T\mathbf{x} \geq 0\},$$

where $\mathbf{Q}$ is an $n \times n$ positive definite matrix and $\mathbf{a} \in \mathbb{R}^n$ is a convex cone.

Downloaded 04/29/19 to 128.208.201.54. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Downloaded 04/29/19 to 128.208.201.54. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

7.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex as well as concave function. Show that f is an affine function; that is, there exist $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} + b$ for any $\mathbf{x} \in \mathbb{R}^n$.

7.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function. Show that for any $\varepsilon > 0$, the function

$$g_\varepsilon(\mathbf{x}) = f(\mathbf{x}) + \varepsilon\|\mathbf{x}\|^2$$

is coercive.

7.5. Let $f : \mathbb{R}^n \to \mathbb{R}$. Prove that f is convex if and only if for any $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{d} \neq \mathbf{0}$, the one-dimensional function $g_{\mathbf{x},\mathbf{d}}(t) = f(\mathbf{x} + t\mathbf{d})$ is convex.

7.6. Prove Theorem 7.13.

7.7. Let $C \subseteq \mathbb{R}^n$ be a convex set. Let f be a convex function over C, and let g be a strictly convex function over C. Show that the sum function $f + g$ is strictly convex over C.

7.8. (i) Let f be a convex function defined on a convex set C. Suppose that f is *not* strictly convex on C. Prove that there exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n (\mathbf{x} \neq \mathbf{y})$ such that f is affine over the segment $[\mathbf{x}, \mathbf{y}]$.

(ii) Prove that the function $f(x) = x^4$ is strictly convex on $\mathbb{R}$ and that $g(x) = x^p$ for $p > 1$ is strictly convex over $\mathbb{R}_+$.

7.9. Show that the log-sum-exp function $f(\mathbf{x}) = \ln\left(\sum_{i=1}^n e^{x_i}\right)$ is *not* strictly convex over $\mathbb{R}^n$.

7.10. Show that the following functions are convex over the specified domain C:

(i) $f(x_1, x_2, x_3) = -\sqrt{x_1x_2} + 2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$ over $\mathbb{R}^3_{++}$.

(ii) $f(\mathbf{x}) = \|\mathbf{x}\|^4$ over $\mathbb{R}^n$.

(iii) $f(\mathbf{x}) = \sum_{i=1}^n x_i \ln(x_i) - \left(\sum_{i=1}^n x_i\right)\ln\left(\sum_{i=1}^n x_i\right)$ over $\mathbb{R}^n_{++}$.

(iv) $f(\mathbf{x}) = \sqrt{\mathbf{x}^T\mathbf{Q}\mathbf{x} + 1}$ over $\mathbb{R}^n$, where $\mathbf{Q} \succeq \mathbf{0}$ is an $n \times n$ matrix.

(v) $f(x_1, x_2, x_3) = \max\{\sqrt{x_1^2 + x_2^2 + 20x_3^2 - x_1x_2 - 4x_2x_3 + 1}, (x_1^2 + x_2^2 + x_1 + x_2 + 2)^2\}$ over $\mathbb{R}^3$.

(vi) $f(x_1, x_2) = (2x_1^2 + 3x_2^2)\left(\frac{1}{2}x_1^2 + \frac{1}{3}x_2^2\right)$.

7.11. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(\mathbf{x}) = \ln\left(\sum_{i=1}^m e^{\mathbf{A}_i\mathbf{x}}\right),$$

where $\mathbf{A}_i$ is the ith row of $\mathbf{A}$. Prove that f is convex over $\mathbb{R}^n$.

7.12. Prove that the following set is a convex subset of $\mathbb{R}^{n+2}$:

$$C = \left\{ \begin{pmatrix} \mathbf{x} \\ y \\ z \end{pmatrix} : \|\mathbf{x}\|^2 \leq yz, \mathbf{x} \in \mathbb{R}^n, y, z \in \mathbb{R}_+ \right\}.$$

7.13. Show that the function $f(x_1, x_2, x_3) = -e^{(-x_1+x_2-2x_3)^2}$ is not convex over $\mathbb{R}^n$.

7.14. Prove that the geometric mean function $f(\mathbf{x}) = \sqrt[n]{\prod_{i=1}^n x_i}$ is concave over $\mathbb{R}^n_{++}$. Is it strictly concave over $\mathbb{R}^n_{++}$?

(iv) Prove that for any $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$

$$\langle\nabla f(\mathbf{x})-\nabla f(\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle \geq \frac{1}{L}\|\nabla f(\mathbf{x})-\nabla f(\mathbf{y})\|^2 \text{ for any } \mathbf{x},\mathbf{y}\in\mathbb{R}^n.$$

7.29. Let $f:\mathbb{R}^n\to\mathbb{R}\cup\{\infty\}$ be an extended real-valued function. Show that f is convex if and only if $\text{epi}(f)$ is convex.

7.30. Show that the support function of the set $S=\{\mathbf{x}\in\mathbb{R}^n:\mathbf{x}^T\mathbf{Q}\mathbf{x}\leq 1\}$, where $\mathbf{Q}\succ 0$, is $\sigma_S(\mathbf{y})=\sqrt{\mathbf{y}^T\mathbf{Q}^{-1}\mathbf{y}}$.

7.31. Let $S=\{\mathbf{x}\in\mathbb{R}^n:\mathbf{a}^T\mathbf{x}\leq b\}$, where $0\neq\mathbf{a}\in\mathbb{R}^n$ and $b\in\mathbb{R}$. Find the support function σ_S.

7.32. Let $p>1$. Show that the support function of $S=\{\mathbf{x}\in\mathbb{R}^n:\|\mathbf{x}\|_p\leq 1\}$ is $\sigma_S(\mathbf{y})=\|\mathbf{y}\|_q$, where q is defined by the relation $\frac{1}{p}+\frac{1}{q}=1$.

7.33. Let $f_0,f_1,\ldots,f_m$ be convex functions over $\mathbb{R}^n$ and consider the perturbation function

$$F(\mathbf{b})=\min_{\mathbf{x}}\{f_0(\mathbf{x}):f_i(\mathbf{x})\leq b_i,i=1,2,\ldots,m\}.$$

Assume that for any $\mathbf{b}\in\mathbb{R}^m$ the minimization problem in the above definition of $F(\mathbf{b})$ has an optimal solution. Show that F is convex over $\mathbb{R}^m$.

7.34. Let $C\subseteq\mathbb{R}^n$ be a convex set and let $\phi_1,\ldots,\phi_m$ be convex functions over C. Let U be the following subset of $\mathbb{R}^m$:

$$U=\{\mathbf{y}\in\mathbb{R}^m:\phi_1(\mathbf{x})\leq y_1,\ldots,\phi_m(\mathbf{x})\leq y_m \text{ for some } \mathbf{x}\in C\}.$$

Show that U is a convex set.

7.35. (i) Show that the extreme points of the unit simplex Δ_n are the unit-vectors $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n$.

(ii) Find the optimal solution of the problem

$$\begin{array}{ll}\max & 57x_1^2+65x_2^2+17x_3^2+96x_1x_2-32x_1x_3+8x_2x_3+27x_1-84x_2+20x_3\\ \text{s.t.} & x_1+x_2+x_3=1\\ & x_1,x_2,x_3\geq 0.\end{array}$$

7.36. Prove that for any $x_1,x_2,\ldots,x_n\in\mathbb{R}_+$ the following inequality holds:

$$\frac{\sum_{i=1}^n x_i}{n}\leq\sqrt{\frac{\sum_{i=1}^n x_i^2}{n}}.$$

7.37. Prove that for any $x_1,x_2,\ldots,x_n\in\mathbb{R}_{++}$ the following inequality holds:

$$\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i}\leq\sqrt{\frac{\sum_{i=1}^n x_i^3}{\sum_{i=1}^n x_i}}.$$

7.38. Let $x_1,x_2,\ldots,x_n>0$ satisfy $\sum_{i=1}^n x_i=1$. Prove that

$$\sum_{i=1}^n\frac{x_i}{\sqrt{1-x_i}}\geq\sqrt{\frac{n}{n-1}}.$$

Downloaded 04/29/19 to 128.208.201.54. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

We can now use CVX to solve the equivalent problem (8.13):

```
cvx_begin
variable z(3)
minimize(d'*z-2*abs(f)'*sqrt(z))
subject to
sum(z)<=1
z>=0
cvx_end
```

The optimal solution is then computed by $y_i = -\text{sgn}(f_i)\sqrt{z_i}$ and then $\mathbf{x} = \mathbf{U}\mathbf{y}$:

```
>> y=-sign(f).*sqrt(z);
>> x=U*y
x =

   -0.2300
   -0.7259
    0.6482
```

■

Exercises

8.1. Consider the problem

$$(\text{P}) \quad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & g(\mathbf{x}) \leq 0 \\ & \mathbf{x} \in X, \end{array}$$

where f and g are convex functions over $\mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$ is a convex set. Suppose that $\mathbf{x}^*$ is an optimal solution of (P) that satisfies $g(\mathbf{x}^*) < 0$. Show that $\mathbf{x}^*$ is also an optimal solution of the problem

$$\begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X. \end{array}$$

8.2. Let $C = B[\mathbf{x}_0, r]$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and $r > 0$ are given. Find a formula for the orthogonal projection operator P_C.

8.3. Let f be a strictly convex function over $\mathbb{R}^m$ and let g be a convex function over $\mathbb{R}^n$. Define the function

$$h(\mathbf{x}) = f(\mathbf{A}\mathbf{x}) + g(\mathbf{x}),$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$. Assume that $\mathbf{x}^*$ and $\mathbf{y}^*$ are optimal solutions of the unconstrained problem of minimizing h. Show that $\mathbf{A}\mathbf{x}^* = \mathbf{A}\mathbf{y}^*$.

8.4. For each of the following optimization problems (a) show that it is convex, (b) write a CVX code that solves it, and (c) write down the optimal solution (by running CVX).

(i)

$$\begin{array}{ll} \min & x_1^2 + 2x_1x_2 + 2x_2^2 + x_3^2 + 3x_1 - 4x_2 \\ \text{s.t.} & \sqrt{2x_1^2 + x_1x_2 + 4x_2^2 + 4} + \frac{(x_1 - x_2 + x_3 + 1)^2}{x_1 + x_2} \leq 6 \\ & x_1, x_2, x_3 \geq 1. \end{array}$$

Downloaded 04/29/19 to 128.208.201.54. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php