The midterm exam will consist of two parts: (I) Linear Least Squares and (II) Quadratic Optimization. In each part, the first question concerns definitions, theorems, and proofs, and the remaining questions are computational. This format is very much like the quizzes. More detailed descriptions of the questions are given below. This is then followed by a list of sample questions.

I. Linear Least Squares

1. Theory Question: For this question you will need to review all of the vocabulary words as well as the theorems for Linear Least Squares. Central focus is on the role of orthogonal projections.
2. Computational questions: You may be asked to compute the solution to a linear system of equations (such as the normal equations), use a given QR factorization to solve the normal equations, solve a linear least squares problem, solve a polynomial fitting problem, and/or compute an orthogonal projection onto a subspace.

II. Quadratic Optimization

4. Theory Question: For this question you will need to review all of the vocabulary words as well as the theorems in Chapter 4 that were covered in class.
5. Computation: You may be asked to compute an eigenvalue decomposition, check if a matrix is positive definite, and/or compute the solution set to a quadratic optimization problem possibly with constraints.

Sample Questions

(I) Linear Least Squares

Question 1:

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, and consider the linear least squares problem

$$\mathcal{LLS} \qquad \min \frac{1}{2}\|Ax-b\|_2^2 \ .$$

a. Show that the matrix A^TA is always positive semi-definite. Provide necessary and sufficient condition on A under which A^TA is positive definite.

b. Show that $\mathrm{Null}(A^TA) = \mathrm{Nul}(A)$ and $\mathrm{Ran}(A^TA) = \mathrm{Ran}(A^T)$.

c. Show using the normal equations that $\mathcal{LLS}$ always has a solution.

d. State and prove a necessary and sufficient condition on the matrix $A \in \mathbb{R}^{m\times n}$ under which $\mathcal{LLS}$ has a unique global optimal solution.

e. Decribe the QR factorization of A and show how it can be used to construct a solution to $\mathcal{LLS}$.

f. If $\mathrm{Null}(A) = \{0\}$, show that $(A^TA)^{-1}$ is well defined and that $P = A(A^TA)^{-1}A^T$ is the orthogonal projection onto $\mathrm{Ran}(A)$ and that

$$\frac{1}{2}\|P(b)-b\|_2^2 = \min \frac{1}{2}\|Ax-b\|_2^2 \ .$$

g. Let $A \in \mathbb{R}^{m\times n}$ be such that $\mathrm{Ran}(A) = \mathbb{R}^m$. Show that the point $\hat{x} := A^T(AA^T)^{-1}b$ is the unique solution to the problem

$$\min \frac{1}{2}\|x\|_2^2 \quad \text{subject to} \quad Ax = b \ .$$

Question 2:

(A) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

a. Compute the orthogonal projection onto $\mathrm{Ran}(A)$.
b. Compute the orthogonal projection onto $\mathrm{Null}(A^T)$.

(B) Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

a. Compute the orthogonal projection onto $\mathrm{Ran}(A)$.
b. Compute the orthogonal projection onto $\mathrm{Null}(A^T)$.

Question 3:

(A) Consider the function

$$f(x_1, x_2, x_3) = \frac{1}{2}[(2x_1 - 4)^2 + (x_1 - x_2)^2 + (3x_2 + x_3)^2].$$

(a) Write this function in the form of the objective function for a linear least squares problem by specifying the matrix A and the vector b.
(b) Describe the solution set of this linear least squares problem.

(B) Find the quadratic polynomial $p(t) = x_0 + x_1 t + x_2 t^2$ that best fits the following data in the least-squares sense:

t	-2	-1	0	1	2
y	2	-10	0	2	1

(C) Consider the problem LLS with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) What are the normal equations for this A and b.
(b) Solve the normal equations to obtain a solution to the problem LLS for this A and b.
(c) Compute the orthogonal projection onto the range of A.
(d) Describe how you would use a QR factorization $AP = Q[R_1 \ R_2]$ to solve the LLS for this A and b.
(e) If $\bar{x}$ solves LLS for this A and b, what is $A\bar{x} - b$?

(II) Quadratic Optimization

Question 4:

Consider the function

$$f(x) = \frac{1}{2}x^T Hx + v^T x,$$

where $H \in \mathbb{R}^{n\times n}$ is symmetric and $v \in \mathbb{R}^n$.

1. What is the eigenvalue decomposition of H?
2. Give necessary and sufficient conditions on H and v for which there exists a solution to the problem $\min_{x\in\mathbb{R}^n} f(x)$. Justify your answer.
3. If H is positive definite, show that there is a nonsingular matrix B such that $H = B^T B$.
4. Let $\hat{x} \in \mathbb{R}^n$ and S be a subspace of $\mathbb{R}^n$. Give necessary and sufficient conditions on H and v for which there exists a solution to the problem
$$\min_{x\in\hat{x}+S} f(x) \ .$$
5. Show that every local solution to the problem $\min_{x\in\mathbb{R}^n} f(x)$ is necessarily a global solution.

Question 5:

(A) Compute the eigenvalue decomposition of the following matrices.

$$(a)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad (b)\ H = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$(c)\ H = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad (d)\ H = \begin{bmatrix} 5 & -1 & -1 & 1 \\ -1 & 4 & 2 & -1 \\ -1 & 2 & 4 & -1 \\ 1 & -1 & -1 & 5 \end{bmatrix}$$

(B) For each of the matrices H and vectors v below determine the optimal value of the quadratic optimization problem. If an optimal solution exists, compute the complete set of optimal solutions.

a.
$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

b.
$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

c.
$$H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

(B) Consider the matrix $H \in \mathbb{R}^{3\times3}$ and vector $v \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Does there exists a vector $u \in \mathbb{R}^3$ such that $f(tu) \xrightarrow{t\uparrow\infty} -\infty$? If yes, construct u.

(C) Consider the linearly constrained quadratic optimization problem

$$\begin{array}{lll} \mathcal{Q}(H, v, A, b) \qquad & \text{minimize} & \frac{1}{2}x^T H x + v^T x \\ & \text{subject to} & Ax = b \ , \end{array}$$

where H, A, v, and b are given by

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \ v = (1,1,1)^T, \ b = (4,2)^T, \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

a. Compute a basis for the null space of A.

b. Solve the problem $\mathcal{Q}(H, v, A, b)$.