Sample Questions

(I) Linear Least Squares

Question 1:

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, and consider the linear least squares problem

$$\mathcal{LLS} \qquad \min \frac{1}{2}\|Ax - b\|_2^2 \ .$$

a. Show that the matrix $A^T A$ is always positive semi-definite and provide necessary and sufficient condition on A under which $A^T A$ is positive definite.

Solution

$x^T A^T Ax = \|Ax\|_2^2 \geq 0 \ \forall\, x \in \mathbb{R}^n$ so $A^T A$ is always positive semi-definite. $A^T A$ is positive definite if and only if $0 < x^T A^T Ax = \|Ax\|_2^2 \ \forall\, x \in \mathbb{R}^n \setminus \{0\}$, or equivalently, $\mathrm{Nul}(A) = \{0\}$.

b. Show that $\mathrm{Nul}(A^T A) = \mathrm{Nul}(A)$ and $\mathrm{Ran}(A^T A) = \mathrm{Ran}(A^T)$.

Solution

The inclusion $\mathrm{Nul}(A) \subset \mathrm{Nul}(A^T A)$ is trivial. Conversely, equality $A^T Ax = 0$ implies $x^T A^T Ax = 0$, i.e. $\|Ax\|_2^2 = 0$, and hence $Ax = 0$. This establishes $\mathrm{Nul}(A^T A) = \mathrm{Nul}(A)$. Finally, an application of the Theorem of the alternative implies $\mathrm{Ran}(A^T A) = \mathrm{Ran}(A^T)$.

c. Show using the normal equations that $\mathcal{LLS}$ always has a solution.

Solution

By part b. the normal equations $A^T Ax = A^T b$ admits a solution. Solutions of the normal equations are exactly the solutions of the LLS.

d. State and prove a necessary and sufficient condition on the matrix $A \in \mathbb{R}^{m\times n}$ under which $\mathcal{LLS}$ has a unique global optimal solution.

Solution

$\mathrm{Nul}(A) = \{0\}$.

e. Decribe the QR factorization of A and show how it can be used to construct a solution to $\mathcal{LLS}$.

Solution

Chaper 3, section 5.2.

f. If $\mathrm{Nul}(A) = \{0\}$, show that $(A^T A)^{-1}$ is well defined and that $P = A(A^T A)^{-1}A^T$ is the orthogonal projection onto $\mathrm{Ran}(A)$ and that

$$\frac{1}{2}\|P(b) - b\|_2^2 = \min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 \ .$$

Solution

By part b), $\mathrm{Nul}(A^T A) = \mathrm{Nul}(A) = \{0\}$, and hence $A^T A$ is invertible.

The claim about the orthogonal projection is from Theorem 3.1 from Chapter 3 and the paragraph after it.

g. Let $A \in \mathbb{R}^{m\times n}$ be such that $\mathrm{Ran}(A) = \mathbb{R}^m$. Show that the point $\hat{x} := A^T(AA^T)^{-1}b$ is the unique solution to the problem

$$\min \frac{1}{2}\|x\|_2^2 \quad \text{subject to} \quad Ax = b\ .$$

Solution

Theorem 4.1 from Chapter 3.

Question 2:

(A) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

a. Compute the orthogonal projection onto $\mathrm{Ran}(A)$.

Solution: *Notice Nul$(A) = \{0\}$. Hence by part f., we have $P_{\mathrm{Ran}(A)} = A(A^TA)^{-1}A^T$. A computation shows*

$$P_{\mathrm{Ran}(A)} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix}.$$

provided you can compute $(A^TA)^{-1}$. On an exam, I would tell you what the inverse is, so you would not have to compute it by hand.

b. Compute the orthogonal projection onto $\mathrm{Null}(A^T)$.

Solution: *Since* $\mathrm{Null}(A^T) = \mathrm{Ran}(A)^\perp$*, the projection onto* $\mathrm{Null}(A^T)$ *is just $I - P_{\mathrm{Ran}(A)}$, where $P_{\mathrm{Ran}(A)}$ is given above:*

$$I - P_{\mathrm{Ran}(A)} = \frac{1}{4}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

(B) Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

a. Compute the orthogonal projection onto $\mathrm{Ran}(A)$.

Solution: *Observe Nul$(A) \neq \{0\}$. Define the matrix*

$$\widehat{A} := \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Observe Ran(A) = Ran($\widehat{A}$) and Nul(A) = {0}. Hence the projection onto Ran(A) equals $\widehat{A}(\widehat{A}^T\widehat{A})^{-1}\widehat{A}^T$. Then a computation shows

$$P_{Ran(A)} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}.$$

b. Compute the orthogonal projection onto $\text{Null}(A^T)$.

Solution:

$\text{Nul}(A^T) = \text{Ran}(A)^\perp$.

$$P_{\text{Nul}(A^T)} = I - P_{\text{Ran}(A)} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Question 3:

A Consider the function

$$f(x_1, x_2, x_3) = \frac{1}{2}[(2x_1 - 4)^2 + (x_1 - x_2)^2 + (3x_2 + x_3)^2].$$

(a) Write this function in the form of the objective function for a linear least squares problem by specifying the matrix A and the vector b.

Solution:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 3 & 1 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

(b) Describe the solution set of this linear least squares problem.

Solution:

$(x_1, x_2, x_3) = (2, 2, -6)$.

(B) Find the quadratic polynomial $p(t) = x_0 + x_1 t + x_2 t^2$ that best fits the following data in the least-squares sense:

$$\begin{array}{c|ccccc} t & -2 & -1 & 0 & 1 & 2 \\ \hline y & 2 & -10 & 0 & 2 & 1 \end{array}.$$

Solution: Write it as an LLS problem where

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -10 \\ 0 \\ 2 \\ 1 \end{bmatrix},$$

and solving the LLS gives us $(x_0, x_1, x_2) = (-3, 1, 1)$.

(C) Consider the problem LLS with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) What are the normal equations for this A and b.

Solution: *The normal equations are $A^TAx = A^Tb$ (see Theorem 2.1 on page 26 of the notes), where*

$$A^TA = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix} \quad \textit{and} \quad A^Tb = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

(b) Solve the normal equations to obtain a solution to the problem LLS for this A and b.

Solution: *The set of all solutions to the normal equations are*

$$x = \frac{1}{4}\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad t \in \mathbb{R}\ .$$

(c) Compute the orthogonal projection onto the range of A.

Solution:

See Question 2(B) (a).

(d) Use the recipe

$$\begin{aligned} AP &= Q[R_1\ R_2] && \text{the general reduced QR factorization} \\ \hat{b} &= Q^Tb && \text{a matrix-vector product} \\ \bar{w}_1 &= R_1^{-1}\hat{b} && \text{a back solve} \\ \bar{x} &= P\begin{bmatrix} R_1^{-1}\hat{b} \\ 0 \end{bmatrix} && \text{a matrix-vector product.} \end{aligned}$$

to solve LLS for this A and b.

Solution: *See discussion on page 34 of the notes.*

(e) If $\bar{x}$ solves LLS for this A and b, what is $A\bar{x} - b$?

Solution:

$$A\bar{x} - b = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

(II) Quadratic Optimization

Question 4:

Consider the function

$$f(x) = \frac{1}{2}x^THx + v^Tx,$$

where $H \in \mathbb{R}^{n\times n}$ is symmetric and $v \in \mathbb{R}^m$.

1. What is the eigenvalue decomposition of H?

   Solution: Theorem 1.1 from Chapter 4.

2. Give necessary and sufficient conditions on H and v for which there exists a solution to the problem $\min_{x\in\mathbb{R}^n} f(x)$. Justify your answer.

   Solution: Theorem 2.1 from Chapter 4.

3. If H is positive definite, show that there is a nonsingular matrix B such that $Q = B^T B$.

   Solution:

   Consider an eigenvalue decomposition of $H = U\Lambda U^T$, where $\Lambda = \text{diag}\{\lambda_1, \cdots, \lambda_n\}$ where λ_i are the eigenvalues of H. Since H is positive definite, we have $\lambda_i > 0$. Set $\Lambda^{\frac{1}{2}} = \text{diag}\{\lambda_1^{\frac{1}{2}}, \cdots, \lambda_n^{\frac{1}{2}}\}$ and $B = D^{\frac{1}{2}} U^T$, then $H = B^T B$, with B nonsingular.

   Solution: Those where H is positive semidefinite and the system $Hx + v = 0$ is solvable.

4. Let $\hat{x} \in \mathbb{R}^n$ and S be a subspace of $\mathbb{R}^n$. Give necessary and sufficient conditions on Q and c for which there exists a solution to the problem

$$\min_{x \in \hat{x} + S} f(x) \ .$$

   Solution:

   Theorem 3.1 from Chapter 4.

5. Show that every local solution to the problem $\min_{x \in \mathbb{R}^n} f(x)$ is necessarily a global solution.

   Solution:

   Theorem 2.1 from Chapter 4. I won't ask you this on the exam.

Question 5:

(A) Compute the eigenvalue decomposition of the following matrices.

$$(a)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad (b)\ H = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$(c)\ H = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad (d)\ H = \begin{bmatrix} 5 & -1 & -1 & 1 \\ -1 & 4 & 2 & -1 \\ -1 & 2 & 4 & -1 \\ 1 & -1 & -1 & 5 \end{bmatrix}$$

Solution:

$$H = UDU^T.$$

(a)

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2-\sqrt{2} \end{bmatrix}$$

(b)

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c)

$$U = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(d)

$$U = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{\sqrt{2}}{2} & 1 \\ -\frac{\sqrt{2}}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(B) For each of the matrices H and vectors v below determine the optimal value in $\mathcal{Q}$. If an optimal solution exists, compute the complete set of optimal solutions.

a.

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:
The eigenvalues are $2, 2 \pm \sqrt{2}$ so H is positive definite. Therefore the unique optimal solution is given by $-H^{-1}v = (-2, 1, -1)^T$.

b.

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:
The characteristic polynomial is $p(\lambda) = \det(H - \lambda I) = \lambda^3 - 2\lambda^2 - 6\lambda + 8$. Sketching the graph shows one negative and two positive eigenvalues. Hence H is indefinite so that the optimal value is $-\infty$.

c.

$$H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

Solution:
The characteristic polynomial is $p(\lambda) = \det(H - \lambda I) = \lambda[\lambda^2 - 8\lambda + 11]$ whose roots are $\lambda = 0,\ 4 \pm \sqrt{5}$. Hence H is positive semi-definite so that the set of all possible optimal solutions is the set of solutions to the equation $Hx + v = 0$ which is

$$x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \quad \forall\, t \in \mathbb{R}.$$

(B) Consider the matrix $H \in \mathbb{R}^{3\times 3}$ and vector $v \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Does there exists a vector $u \in \mathbb{R}^3$ such that $f(tu) \overset{t\uparrow\infty}{\longrightarrow} -\infty$? If yes, construct u.

Solution:

The eigenvalues show that H is positive semi-definite with one zero eigenvalue. But the system $Hx + v$ is inconsistent, so no optimal solution exists. The vector $u = (-6, 1, 2)^T$ lies in the null-space of H, and $f(tu) = -4t$. Hence as $t \uparrow \infty$, $f(tu) \downarrow -\infty$.

(C) Consider the linearly constrained quadratic optimization problem

$$\mathcal{Q}(H, g, A, b) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Hx + v^T x \\ \text{subject to} & Ax = b \ , \end{array}$$

where H, A, v, and b are given by

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \ v = (1, 1, 1)^T, \ b = (4, 2)^T, \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

a. Compute a basis for the null space of A.
   Solution: A basis of $\text{Nul}(A)$ is $(1, 0, -1)^T$.
b. Solve the problem $\mathcal{Q}(H, g, A, b)$.
   Solution: Recall that the solution must be of the form

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

since the vector $(1, 1, 1)^T$ solves $Ax = b$ and the vector $(1, 0, -1)$ spans the null space of A. Hence this is just a one dimensional problem in t which is solved by taking $t = \frac{1}{2}$.