Math 408A: Linear Algebra Review

Matrices in $\mathbb{R}^{m\times n}$

$A \in \mathbb{R}^{m\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

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columns

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$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

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Matrix Vector Multiplication

A column space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

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$$= x_1\, a_{\bullet 1} + x_2\, a_{\bullet 2} + \cdots + x_n\, a_{\bullet n}$$

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$$= x_1\, a_{\bullet 1} \;+\; x_2\, a_{\bullet 2} \;+\; \cdots \;+\; x_n\, a_{\bullet n}$$

A linear combination of the columns.

The Range of a Matrix

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Range of A

$$\mathrm{Ran}(A) \; = \; \{Ax \;|x \in \mathbb{R}^n\}$$

$$\mathrm{Ran}(A) = \text{ the linear span of the columns of } A$$

A Couple of Special Subspaces

Let $v_1, \dots, v_k \in \mathbb{R}^n$.

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- The linear span of $v_1, \ldots, v_k$:

$$\mathrm{Span}(v_1, \ldots, v_k) = \{\xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_k v_k \mid \xi_1, \ldots, \xi_k \in \mathbb{R}\}$$

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Dot product of two vectors $x^T y = \sum_{i=1}^{n} x_i y_i$.

- The subspace orthogonal to $v_1, \ldots, v_k$:

$$\{v_1, \ldots, v_k\}^\perp = \left\{ z \in \mathbb{R}^n \,\middle|\, z^T v_i = 0,\ i = 1, \ldots, k \right\}$$

Matrix Vector Multiplication

A row space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1\bullet}^T x \\ a_{2\bullet}^T x \\ \vdots \\ a_{m\bullet}^T x \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i} x_i \\ \sum_{i=1}^n a_{2i} x_i \\ \vdots \\ \sum_{i=1}^n a_{mi} x_i \end{bmatrix}$$

The dot product of x with the rows of A.

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$$\mathrm{Nul}(A) \;=\; \text{subspace orthogonal to the rows of } A$$

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\mathrm{Nul}(A) &= \text{subspace orthogonal to the rows of } A \\
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Fundamental Theorem of the Alternative:

$$\mathrm{Nul}(A) = \mathrm{Ran}(A^T)^{\perp} \qquad \mathrm{Ran}(A) = \mathrm{Nul}(A^T)^{\perp}$$

Matrix-matrix multiplication

For matrices $A \in \mathbb{R}^{m\times n}$ and $A \in \mathbb{R}^{n\times k}$ the product $AB \in \mathbb{R}^{m\times k}$ is defined by

$$(AB)_{ij} = a_{i\bullet}^T b_{\bullet j}.$$

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Equivalently

$$AB = [AB_{\bullet 1} \quad AB_{\bullet 2} \quad \ldots \quad AB_{\bullet k}]$$

Block Structured Matrices

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

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where

$$B = \begin{bmatrix} 3 & -4 & 1 \\ 2 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

Multiplication of Block Structured Matrices

Consider the matrix product AM, where

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Can we exploit the structure of A?

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Multiplication of Block Structured Matrices

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Multiplication of Block Structured Matrices

$$AM = \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} BX + Y \\ CY \end{bmatrix}$$

Multiplication of Block Structured Matrices

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Multiplication of Block Structured Matrices

$$
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AM &= \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} BX + Y \\ CY \end{bmatrix} \\
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&= \begin{bmatrix} 4 & -12 \\ 6 & 15 \\ -3 & -2 \\ 0 & 1 \\ -4 & -1 \end{bmatrix}.
\end{aligned}
$$

Solving Systems of Linear equations

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
Find all solutions $x \in \mathbb{R}^n$ to the system $Ax = b$.

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The set of solutions is either empty, a single point, or an infinite set.

If a solution $x_0 \in \mathbb{R}^n$ exists, then the set of solutions is given by

$$x_0 \; + \; \mathrm{Nul}(A) \; .$$

Gaussian Elimination and the 3 Elementary Row Operations

We solve the system $Ax = b$ by transforming the augmented matrix

$$[\,A \mid b\,]$$

into upper echelon form using the three elementary row operations.

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These elementary row operations can be interpreted as multiplying the augmented matrix on the left by a special matrix.

Exchange and Permutation Matrices

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Multiplying any $4 \times n$ matrix on the left by the exchange matrix

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$$AN = \begin{bmatrix} a_{1\bullet}N \\ a_{2\bullet}N \\ \vdots \\ a_{m\bullet}N \end{bmatrix}$$

Gaussian Elimination Matrices

The key step in Gaussian elimination is to transform a vector of the form

$$\begin{bmatrix} a \\ \alpha \\ b \end{bmatrix},$$

where $a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$, into one of the form

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This can be accomplished by left matrix multiplication as follows.

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Note that a Gaussian elimination matrix and its inverse are both lower triangular matrices.

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Transformation to echelon
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Gauss-Jordan Elimination, or Pivot Matrices

What happens in the following multiplication?

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