Math 408 Homework Set 4

(1) Use the "delta method" (as we did in class for the Linear Least Squares function) to compute the gradient and the Hessian of the following functions.
  (a) $f(x) := \frac{1}{2}\|Ax - b\|_2^2$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.
  (b) $f(x) := \frac{1}{2}\|F(x)\|_2^2$, where $F : \mathbb{R}^n \to \mathbb{R}^m$ is such that all of its component functions $F_i : \mathbb{R}^n \to \mathbb{R}$ twice differentiable.
  (c) $f(x) := \frac{1}{2}(x - \bar{x})^T H(x - \bar{x}) + g^T(x - \bar{x})$, where $H \in \mathbb{R}^{n\times n}$ is symmetric and $\bar{x}, g \in \mathbb{R}^n$.

(2) Compute the gradient and Hessian of each of the following functions using first-order and second-order partial derivatives (if they exist).
  (a) $f(x) = x_1^2 - 4x_1 + 2x_2^2 + 7$
  (b) $f(x) = e^{-\|x\|^2}$
  (c) $f(x) = x_1^2 - 2x_1x_2 + \frac{1}{3}x_2^3 - 8x_2$
  (d) $f(x) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$
  (e) $f(x) = x_1^4 + 16x_1x_2 + x_2^4$
  (f) $f(x) = (1 - x_1)^2 + \sum_{j=1}^{n-1} 10^j(x_j - x_{j+1}^2)^2$ (The Rosenbrock function)
  (g) $f(x) = 3x_1^2 + x_1x_2x_3$
  (h) $f(x) = 2\cos(x_1)\sin(x_2x_3)$
  (i) $f(x) = \ln[\exp(x_1^2) + \exp(x_2^2) + \exp(x_3^2)]$
  (j) $f(x) = 1/\sqrt{1 + (x_2x_3)^2}$
  (k) $f(x, y) = 5x^2 + 2xy + y^2 - x + 2y + 3$
  (l) $f(x, y) = \begin{cases} (x + 2y + 1)^8 - \log((xy)^2), & \text{if } 0 < x,\ 0 < y, \\ +\infty, & \text{otherwise.} \end{cases}$
  (m) $f(x, y) = 4e^{3x-y} + 5e^{x^2+y^2}$
  (n) $f(x, y) = \begin{cases} x + \frac{2}{x} + 2y + \frac{4}{y}, & \text{if } 0 < x,\ 0 < y, \\ +\infty, & \text{otherwise.} \end{cases}$

(3) A critical point of a function $f : \mathbb{R}^n \to \mathbb{R}$ is any point x at which $\nabla f(x) = 0$. Compute all of the critical points of the functions in Problem (2). If no critical point exists, explain why.