Math 408 Homework Set 3

This homework set will focus on the optimization problem

$$\mathcal{Q} \qquad \min_{x\in\mathbb{R}^n} \frac{1}{2}x^T Hx + g^T x \ ,$$

where $H \in \mathbb{R}^{n\times n}$ is symmetric and $g \in \mathbb{R}^n$.

(1) Each of the following functions can be written in the form $f(x) = \frac{1}{2}x^T Hx + g^T x$ with H symmetric. For each of these functions what are H and g
  (a) $f(x) = x_1^2 - 4x_1 + 2x_2^2 + 7$
  (c) $f(x) = x_1^2 - 2x_1x_2 + \frac{1}{2}x_2^2 - 8x_2$
  (d) $f(x) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$
  (e) $f(x) = x_1^2 + 16x_1x_2 + 4x_2x_3 + x_2^2$

(2) For $a, b \in \mathbb{R}$, consider the matrix

$$H = \begin{bmatrix} 2 & a & 0 \\ a & 2 & b \\ 0 & b & 2 \end{bmatrix} .$$

  (a) Compute the eigenvalues of H as functions of a and b.
  (b) For what values of a and b is H positive definite.
  (c) For what values of a and b is H positive semi-definite.
  (d) For what values of a and b is H negative semi-definite.
  (e) For what values of a and b is H negative definite.
  (f) For what values of a and b is H indefinite.

(3) Consider the matrix

$$H = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 9 & 3 \\ 2 & 3 & 4 \end{bmatrix} .$$

  (a) Compute the eigenvalues of H.
  (b) Compute and orthonormal basis of eigenvectors for H.
  (c) Compute the eigenvalue decomposition of H.

(4) For each of the matrices H and vectors g below determine the optimal value in $\mathcal{Q}$. If an optimal solution exists, compute the complete set of optimal solutions.
  (a)

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} .$$

  (b)

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} .$$

  (c)

$$H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} .$$

(5) Consider the matrix $H \in \mathbb{R}^{3\times 3}$ and vector $g \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Does there exists a vector $u \in \mathbb{R}^3$ such that $f(tu) \xrightarrow{t\uparrow\infty} -\infty$? If yes, construct u.

(6) Let $H \in \mathbb{R}^{n\times n}$ be symmetric with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. For $\gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma_1 \leq \gamma_2$, show that $\gamma_1 \leq \lambda_j \leq \gamma_2$ for all $j = 1, 2, \ldots, n$ if and only if

$$\gamma_1 \|u\|_2^2 \leq u^T H u \leq \gamma_2 \|u\|_2^2$$

for all $u \in \mathbb{R}^n$.