This exam will consist of three parts: (I) Linear Least Squares, (II) Quadratic Optimization, and (III) Optimality Conditions and Line Search Methods. The topics covered on the first two parts ((I) Linear Least Squares and (II) Quadratic Optimization) are identical in content to the two parts of the midterm exam. Please use the midterm exam study guide to prepare for these questions. A more detailed description of the third part of the final exam is given below.

III Optimality Conditions and Line search methods.

1 Theory Question: For this question you will need to review all of the vocabulary words as well as the theorems on the weekly guides for *Elements of Multivariable Calculus, Optimality Conditions for Unconstrained Problems*, and *Line search methods*. You may be asked to provide statements of first- and second-order optimality conditions for unconstrained problems. In addition, you may be asked about the role of convexity in optimization, how it is detected, as well as first- and second-order conditions under which it is satisfied. You may be asked to describe what a descent direction is, as well as, what the Newton and the Steepest descent directions are. You need to know what the backtracking line-search is, as well as the convergence guarantees of the line search methods.

2 Computation: All the computation needed for Linear Least Squares and Quadratic Optimization is fair game for this question. Mostly, however, you will be asked to compute gradients and Hessians, locate and classify stationary points for specific optimizations problems, as well as test for the convexity of a problem. You may also be asked to write out the iterates generated by a line search method on a simple function.

Sample Questions

(III) Optimality Conditions and Line Search Methods

Question 1: Theory Question

1. State the first- and second-order conditions for optimality for the following two problems:
   (a) Linear least squares: $\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax-b\|_2^2$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.
   (b) Quadratic Optimization: $\min_{x\in\mathbb{R}^n} \frac{1}{2}x^TQx + g^Tx$, where $Q \in \mathbb{R}^{n\times n}$ is symmetric and $g \in \mathbb{R}^n$.
2. What is an H-orthogonal vector pair, and why must the vectors in this pair be linearly independent, whenever they are nonzero?
3. Provide necessary and sufficient conditions under which a symmetric matrix X can be written as $X = A^TA$ for some matrix A.
4. State the second-order necessary and sufficient optimality conditions for the problem $\min_{x\in\mathbb{R}^n} f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable.
5. State first- and second-order necessary and sufficient conditions for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be convex.
6. What is the relation between local minimizers, global minimizers, and critical points of differentiable convex functions?
7. What is a line search methods?
8. Define direction of descent, steepest descent direction, and the Newton direction.

9. State what is the backtracking line search.

Question 2: Computation

1. If f_1 and f_2 are convex functions mapping $\mathbb{R}^n$ into $\mathbb{R}$, show that $f(x) := \max\{f_1(x), f_2(x)\}$ is also a convex function. [**Hint**:Use the definition of convexity.]

2. Use the delta method to compute the gradient of the function $f(x) = \frac{1}{2}x^T Hx + v^T x$, where H is symmetric.

3. Let $Q \in \mathbb{R}^{n\times m}$ be symmetric and $v \in \mathbb{R}^m$. Compute the gradient of the function

$$f(x) := \|Qx\|^2 + v^T x.$$

4. A critical point of a function $f : \mathbb{R}^n \to \mathbb{R}$ is any point x at which $\nabla f(x) = 0$. Compute all of the critical points of the following functions. If no critical point exists, explain why.

   (a) $f(\bar{x}) = x_1^2 - 4x_1 + 2x_2^2 + 7$
   (b) $f(x) = e^{-\|x\|^2}$
   (c) $f(x) = x_1^2 - 2x_1x_2 + \frac{1}{3}x_2^3 - 8x_2$
   (d) $f(x) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$

5. Show that the functions

$$f(x_1, x_2) = x_1^2 + x_2^3, \quad \text{and} \quad g(x_1, x_2) = x_1^2 + x_2^4$$

   both have a critical point at $(x_1, x_2) = (0, 0)$ and that their associated Hessians are positive semi-definite. Then show that $(0, 0)$ is a local (global) minimizer for g but is not a local minimizer for f.

6. Find the local minimizers for the following functions if they exist:

   (a) $f(x) = x^2 + \cos x$
   (b) $f(x_1, x_2) = x_1^2 - 4x_1 + 2x_2^2 + 7$
   (c) $f(x_1, x_2) = e^{-(x_1^2+x_2^2)}$
   (d) $f(x_1, x_2, x_3) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$

7. Determine which function in question (6) are convex and what that means for your answer in question (6).

8. Compute the gradients and Hessians of the functions

   (a) $f(x) = \ln\left(e^{x_1^2} + e^{x_2^2} + e^{x_3^2}\right)$
   (b) $f(x) = e^{x_1}\sin(x_1)\sin(x_2)$
   (c) $f(x) = \ln\Big(2 + \sin(x_1) + \sin(x_2)\Big)$

9. Run the steepest descent method on the function $f(x, y) = \frac{1}{2}x^2 + y^2$ starting at the point $x_1 = (2, 2)$ and using fixed step sizes $t_k = \frac{1}{2}$. Output the points x_1, x_2, x_3.