Math 408  

Homework Set 5

(1) Show that the functions

\[ f(x_1, x_2) = x_1^2 + x_2^2, \quad \text{and} \quad g(x_1, x_2) = x_1^2 + x_2^4 \]

both have a critical point at \((x_1, x_2) = (0, 0)\) (i.e. \(\nabla f(x_1, x_2) = \nabla g(x_1, x_2) = 0\)) and that their associated hessians are positive semi-definite. Then show that \((0, 0)\) is a local (global) minimizer for \(g\) and not for \(f\).

(2) Find the local minimizers and maximizers for the following functions if they exist:

(a) \(f(x) = x^2 + \cos x\)
(b) \(f(x_1, x_2) = x_1^2 - 4x_1 + 2x_2^2 + 7\)
(c) \(f(x_1, x_2) = e^{-(x_1^2+x_2^2)}\)
(d) \(f(x_1, x_2, x_3) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2\)

(3) Which of the functions in problem 2 above are convex and why?

(4) If \(f : \mathbb{R}^n \to \mathbb{R} = \mathbb{R} \cup \{+\infty\}\) is convex, show that the sets \(\text{lev}_f(\alpha) = \{x : f(x) \leq \alpha\}\) are convex sets for every \(\alpha \in \mathbb{R}\). Let \(h(x) = x^3\). Show that the sets \(\text{lev}_h(\alpha)\) are convex for all \(\alpha\), but the function \(h\) is not itself a convex function.

(5) Show that each of the following functions is convex.

(a) \(f(x) = e^{-x}\)
(b) \(f(x_1, x_2, \ldots, x_n) = e^{-(x_1+x_2+\ldots+x_n)}\)
(c) \(f(x) = \|x\|\)

(6) If \(f_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1, 2\) are convex, show that \(f(x) = \max\{f_1(x), f_2(x)\}\) is a convex function.

(7) Let \(A \in \mathbb{R}^{m \times n}\) and \(b \in \mathbb{R}^m\), and suppose that \(f : \mathbb{R}^m \to \mathbb{R}\) is convex. Show that \(h(x) = f(Ax + b)\) is convex.

(8) Consider the functions

\[ f(x) = \frac{1}{2}x^TQx - c^Tx \]

and

\[ f_i(x) = \frac{1}{2}x^TQx - c^Tx + t\phi(x), \]

where \(t > 0\), \(Q \in \mathbb{R}^{n \times n}\) is positive semi-definite, \(c \in \mathbb{R}^n\), and \(\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) is given by

\[ \phi(x) = \begin{cases} -\sum_{i=1}^n \ln x_i, & \text{if } x_i > 0, \ i = 1, 2, \ldots, n, \\ +\infty, & \text{otherwise}. \end{cases} \]

(a) Show that \(\phi\) is a convex function.
(b) Show that both \(f\) and \(f_i\) are convex functions.
(c) Show that for all \(t > 0\) the solution to the problem \(\min f_i(x)\) always exists and is unique.
(d) Let \(x(t)\) denote the unique solution to \(\min f_i(x)\) for \(t > 0\). Show that \(x(t) > 0\).
(e) Define \(u(t) = t\nabla \phi(x(t))\). If there exists a sequence \(t_i \downarrow 0\) and a point \((\bar{x}, \bar{u})\) such that \((x(t_i), u(t_i)) \to (\bar{x}, \bar{u})\), show that \(\bar{x}\) solves \(\min_{0 \leq x} f(x)\).

(9) Show that each of the following functions is convex.

(a) \(f(x, y) = 5x^2 + 2xy + y^2 - x + 2y + 3\)
(b) \(f(x, y) = \begin{cases} (x + 2y + 1)^8 - \log((xy)^2), & \text{if } 0 < x, \ 0 < y, \\ +\infty, & \text{otherwise}. \end{cases} \)
(c) \(f(x, y) = 4e^{3x-y} + 5e^{x^2+y^2}\)
(d) \(f(x, y) = \begin{cases} x + \frac{2}{x} + 2y + \frac{4}{y}, & \text{if } 0 < x, \ 0 < y, \\ +\infty, & \text{otherwise}. \end{cases} \)

(10) Compute the global minimizers of the functions given in the previous problem if they exist.