This homework set will focus on the optimization problem

\[ Q = \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T H x + g^T x , \]

where \( H \in \mathbb{R}^{n \times n} \) is symmetric and \( g \in \mathbb{R}^n \).

(1) Each of the following functions can be written in the form \( f(x) = \frac{1}{2} x^T H x + g^T x \) with \( H \) symmetric. For each of these functions what are \( H \) and \( g \)

(a) \( f(x) = x_1^2 - 4x_1 + 2x_2^2 + 7 \)
(c) \( f(x) = x_1^2 - 2x_1x_2 + \frac{1}{2}x_2^2 - 8x_2 \)
(d) \( f(x) = 2(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2 \)
(e) \( f(x) = x_1^2 + 16x_1x_2 + 4x_2x_3 + x_2^2 \)

(2) For \( a, b \in \mathbb{R} \), consider the matrix

\[
H = \begin{bmatrix}
2 & a & 0 \\
a & 2 & b \\
0 & b & 2
\end{bmatrix}.
\]

(a) Compute the eigenvalues of \( H \) as functions of \( a \) and \( b \).
(b) For what values of \( a \) and \( b \) is \( H \) positive definite.
(c) For what values of \( a \) and \( b \) is \( H \) positive semi-definite.
(d) For what values of \( a \) and \( b \) is \( H \) negative semi-definite.
(e) For what values of \( a \) and \( b \) is \( H \) negative definite.
(f) For what values of \( a \) and \( b \) is \( H \) indefinite.

(3) Consider the matrix

\[
H = \begin{bmatrix}
4 & 3 & 2 \\
3 & 9 & 3 \\
2 & 3 & 4
\end{bmatrix}.
\]

(a) Compute the eigenvalues of \( H \).
(b) Compute and orthonormal basis of eigenvectors for \( H \).
(c) Compute the eigenvalue decomposition of \( H \).

(4) For each of the matrices \( H \) and vectors \( g \) below determine the optimal value in \( Q \). If an optimal solution exists, compute the complete set of optimal solutions.

(a)

\[
H = \begin{bmatrix}
2 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.
\]

(b)

\[
H = \begin{bmatrix}
2 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & -2
\end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.
\]

(c)

\[
H = \begin{bmatrix}
5 & 2 & -1 \\
2 & 1 & -1 \\
-1 & -1 & 2
\end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.
\]
(5) Consider the matrix $H \in \mathbb{R}^{3 \times 3}$ and vector $g \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix}
1 & 4 & 1 \\
4 & 20 & 2 \\
1 & 2 & 2
\end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}.$$ 

Does there exists a vector $u \in \mathbb{R}^3$ such that $f(tu) \uparrow \infty$? If yes, construct $u$.

(6) Let $H \in \mathbb{R}^{n \times n}$ be symmetric with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. For $\gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma_1 \leq \gamma_2$, show that $\gamma_1 \leq \lambda_j \leq \gamma_2$ for all $j = 1, 2, \ldots, n$ if and only if

$$\gamma_1 \|u\|_2^2 \leq u^T Hu \leq \gamma_2 \|u\|_2^2$$

for all $u \in \mathbb{R}^n$. 