Slope and variational geometry in optimization

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A.W. Tucker Prize session

AFOSR YIP

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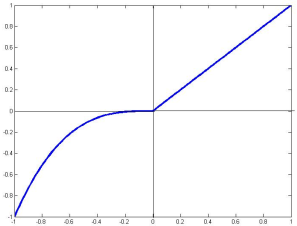
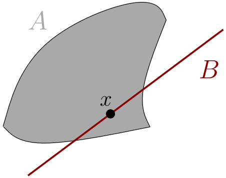


Origin is critical

Method of alternating projections

Common problem:

Given sets $A, B \subset \mathbf{R}^n$, find some point $x \in A \cap B$.



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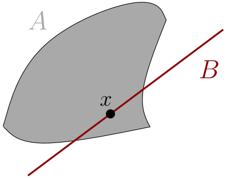

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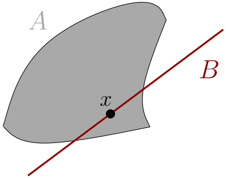

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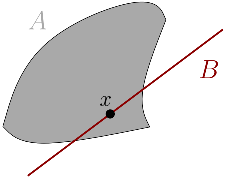

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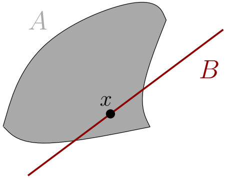

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Eg 2 (Low-rank SDP):

$$\{X \succeq 0 : \operatorname{rank} X \leq r\} \bigcap \{X : \mathcal{A}(X) = b\}.$$

Method of alternating projections

Method of alternating projections (von Neumann '33):

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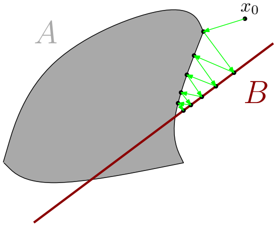

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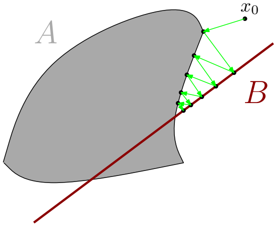

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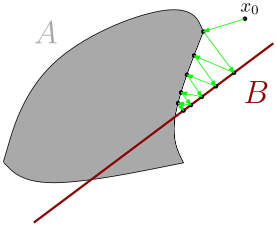

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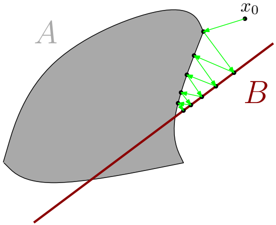

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If $|\nabla\psi| \geq \frac{1}{\kappa}$ near $A \cap B$, then get local linear convergence!

(D-Ioffe-Lewis '13)

Slope and error bounds

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Lemma (Error bound)

The following are equivalent.

Non-criticality:

$$|\nabla f| \ge \frac{1}{\kappa}.$$

Error-bound:

$$\operatorname{dist}(x, [f \le r]) \le \kappa(f(x) - r), \quad \text{when } r \in (a, f(x)).$$

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Fréchet subdifferential:

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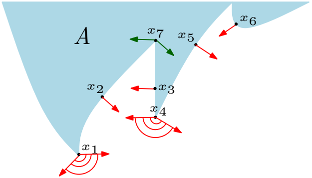
Normal cone $N_A(x) = \partial \delta_A(x)$ appears:



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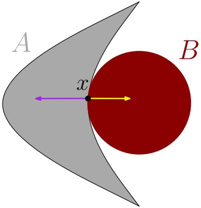

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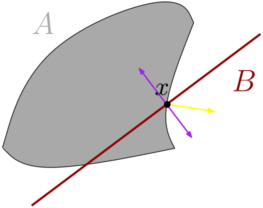

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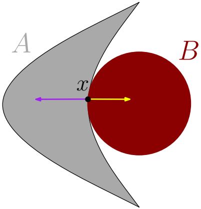

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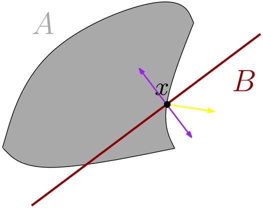

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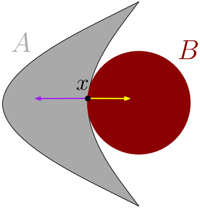

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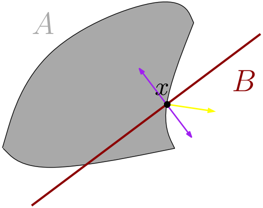

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Extremal principle (Mordukhovich):

Transversality $\implies$ $(A + a) \cap B \neq \emptyset$ for small a.

Subdifferentials of eigenvalue functions

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(Ball, Daniilidis, Davis, Lewis, Malick, Sendov, Šilhavý, Sylvester, von Neumann, ...)

Subdifferential formula

Main tool (Lewis '99): $V \in \partial(f \circ \lambda)(X)$ exactly when

$$\left\{ \begin{array}{c} V = U(\operatorname{Diag} v)\,U^T \\ X = U(\operatorname{Diag} \lambda(X))\,U^T \end{array} \right\} \text{ with } U \in \mathbb{O}^n \text{ and } v \in \partial f(\lambda(X)).$$

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Nonconvex proof (D-Kempton '14):

$$\boxed{(f \circ \lambda)_\alpha = f_\alpha \circ \lambda}$$

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$$f_\alpha(x) = \inf_y \left\{ f(y) + \frac{1}{2\alpha}|y - x|^2 \right\} \qquad \text{is the Moreau envelope.}$$

Active manifolds

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Definition (Active manifolds)

Manifold $\mathcal{M} \subset \mathbf{R}^n$ is active for f at $\bar{x}$ provided $f\big|_{\mathcal{M}}$ is smooth and

$$\left.\begin{array}{c} x_i \to \bar{x}, v_i \to 0 \\ v_i \in \partial f(x_i) \end{array}\right\} \Longrightarrow x_i \in \mathcal{M} \text{ for all large } i$$

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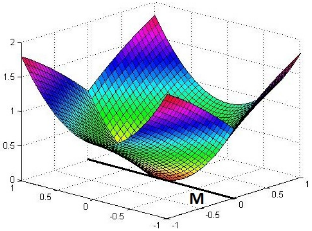

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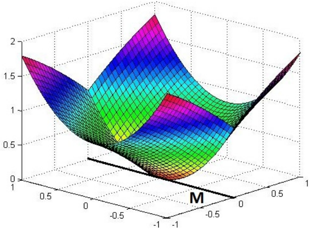

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• **Common**: appear from regularizers (l_∞, $l_{1,2}$, $\|\cdot\|_*$, TV), and are generic for semi-algebraic problems (D-Ioffe-Lewis '14).

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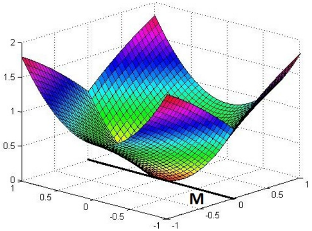

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- **Computation**: algorithm acceleration on $\mathcal{M}$. (Fadili at ISMP)

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Definition (Active manifolds)

Manifold $\mathcal{M} \subset \mathbf{R}^n$ is active for f at $\bar{x}$ provided $f|_{\mathcal{M}}$ is smooth and

$$\left.\begin{array}{c} x_i \to \bar{x},\, v_i \to 0 \\ v_i \in \partial f(x_i) \end{array}\right\} \Longrightarrow x_i \in \mathcal{M} \text{ for all large } i$$

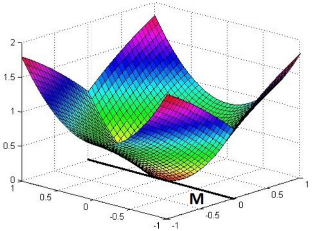


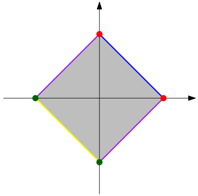
Figure: $f(x,y) = y^2 - x^2 + |x|$, $\quad \mathcal{M} = \{0\} \times \mathbf{R}$

Introduced by Wright '93, Lewis '04.

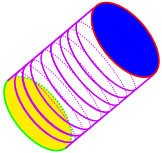
- **Common**: appear from regularizers (l_∞, $l_{1,2}$, $\|\cdot\|_*$, TV), and are generic for semi-algebraic problems (D-Ioffe-Lewis '14).
- **Computation**: algorithm acceleration on $\mathcal{M}$. (Fadili at ISMP)
- **Sensitivity**: only the restriction $f|_{\mathcal{M}}$ matters (D-Lewis '14).

Spectral active sets

Example:

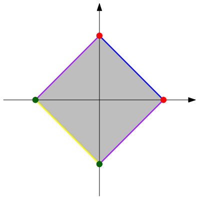


$|x| + |y| \leq 1$

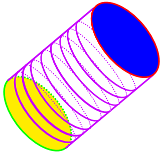


$|\lambda_1(X)| + |\lambda_2(X)| \leq 1$

Example:



$|x| + |y| \leq 1$

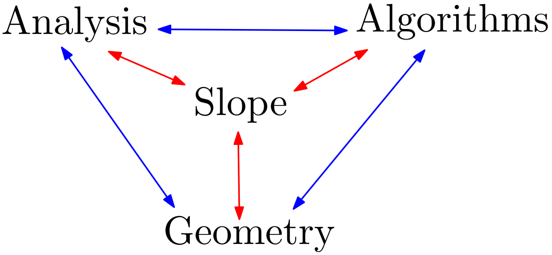


$|\lambda_1(X)| + |\lambda_2(X)| \leq 1$

Spectral active sets (Daniilidis-D-Lewis '14):

$\mathcal{M}$ active manifold at $\lambda(X)$ for f

$\iff \lambda^{-1}(\mathcal{M})$ active manifold at X for $f \circ \lambda$.


Analysis
Algorithms
Slope
Geometry

Thank you.