Variational analysis with smooth substructure

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Goals

- Intuitive notion of identifiable sets.
  - Existence, calculus.
  - Properties of identifiable sets.
- Semi-algebraic geometry.
  - Identifiable manifolds exist generically.
  - Size of semi-algebraic subdifferential graphs.

Preliminaries (Subdifferentials)

For a function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$, a vector $\bar{v}$ is a Frechét subgradient at $\bar{x}$, denoted $\bar{v} \in \hat{\partial} f(\bar{x})$, if

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$\bar{x}$ is a critical point of f if $0 \in \partial f(\bar{x})$.

- For convex f, critical points are global minimizers.
- If f is $\mathbf{C}^1$-smooth, criticality reduces to the classical condition $\nabla f(x) = 0$.

Motivation (Sensitivity Analysis)

Consider the perturbed functions

$$f_v(x) = f(x) - \langle v, x \rangle. \qquad [\textit{For simplicity}],$$

and suppose $\bar{x}$ is critical for $f_{\bar{v}}$, that is $\bar{v} \in \partial f(\bar{x})$.

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Sensitivity question: How do critical points of f_v, near $\bar{x}$, behave as v varies near $\bar{v}$?
Thus given $\bar{v} \in \partial f(\bar{x})$, we want to understand how solutions x_v of

$$v \in \partial f(x),$$

vary, as we perturb v near $\bar{v}$.

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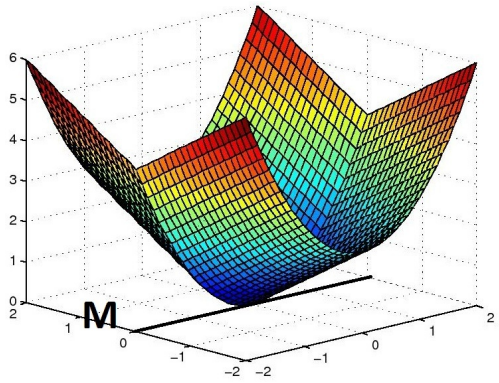

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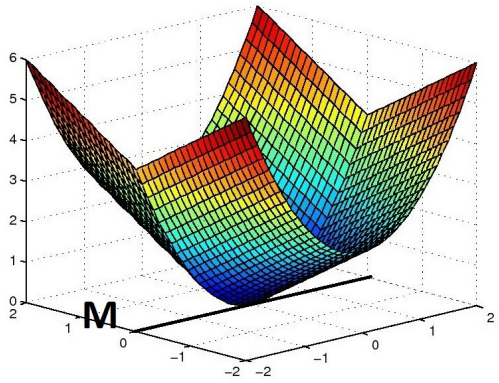

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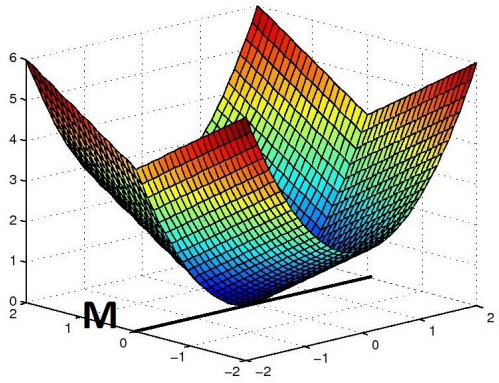

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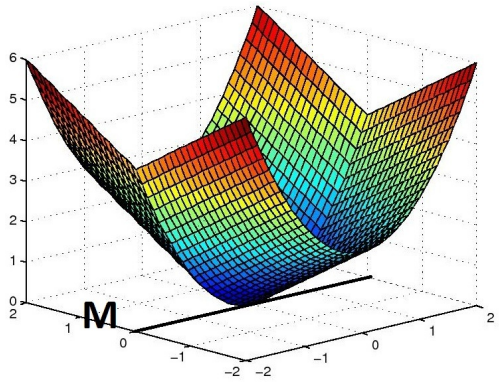

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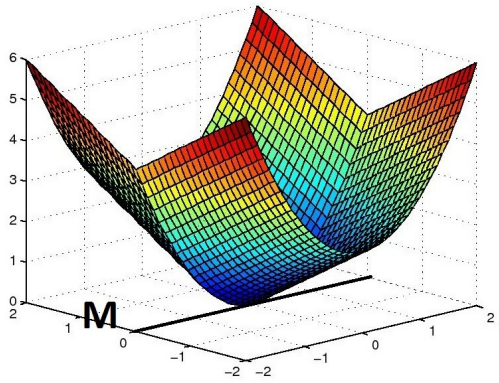

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- Only the restriction $f\big|_M$ matters!
- Goal: Look for small, well-behaved sets capturing only the essential information.

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$$v \in \partial f(x).$$

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$$\left.\begin{array}{c} x_i \to \bar{x}, v_i \to \bar{v} \\ v_i \in \partial f(x_i) \end{array}\right\} \Longrightarrow x_i \in M \text{ for all large } i,$$

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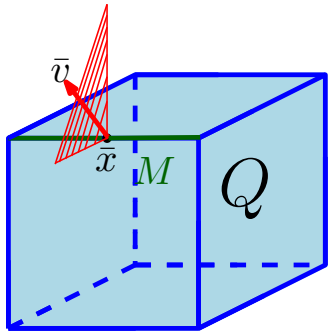

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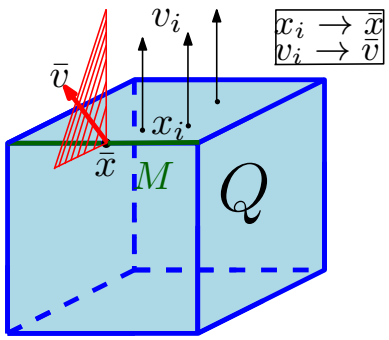

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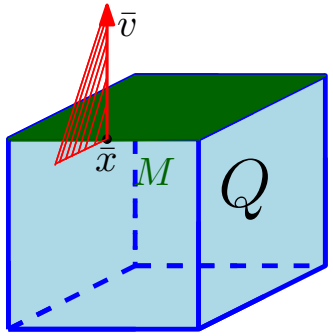

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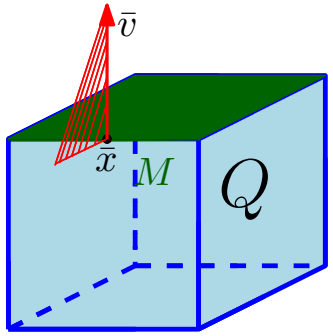

In this case $M = \operatorname{argmax}_{x \in Q} \langle \bar{v}, x \rangle$.

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Suppose M is an identifiable set at $\bar{x}$ for $0 \in \hat{\partial} f(\bar{x})$.

- *$\bar{x}$ is a (strict) local minimizer of $f \iff \bar{x}$ is a (strict) local minimizer of f on M.*
- *f grows quadratically near $\bar{x} \iff f$ grows quadratically on M near $\bar{x}$.*

Subgradient dynamical systems

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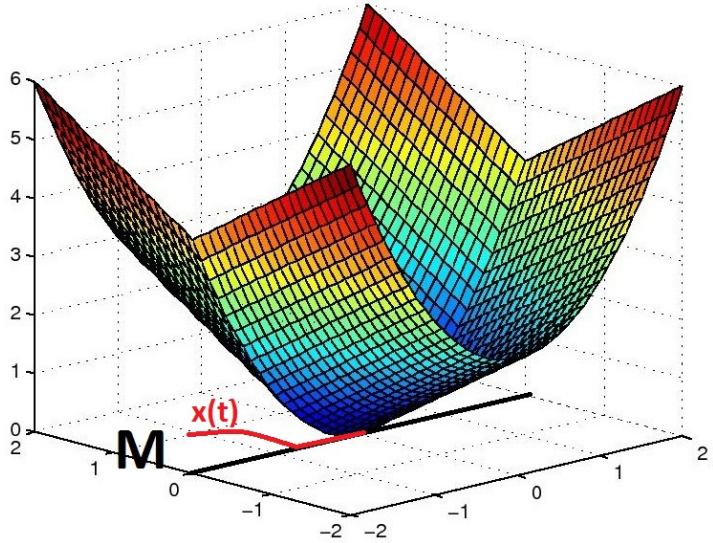

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An identifiable set M at $\bar{x}$ for $\bar{v}$ is locally minimal if

$$M' \text{ identifiable at } \bar{x} \text{ for } \bar{v} \Longrightarrow M \subset M', \text{ locally near } \bar{x}.$$

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- convex polyhedra $[M = \operatorname{argmax}_{x \in Q} \langle \bar{v}, x \rangle]$,
- piecewise quadratic functions,
- max-type functions: $f(x) = \max\{g_1(x), \ldots, g_k(x)\}$ for $\mathbf{C}^1$-smooth g_i,
- fully amenable functions: $f(x) = g(F(x))$ where
  1. F is $\mathbf{C}^2$-smooth,
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A strong chain rule is available for composite functions

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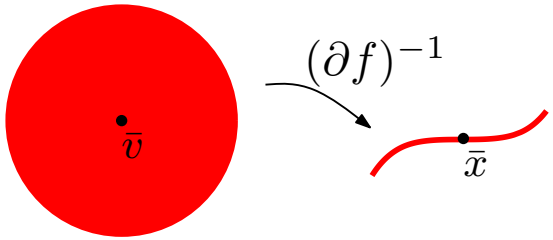

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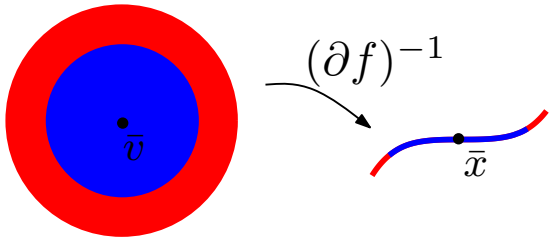

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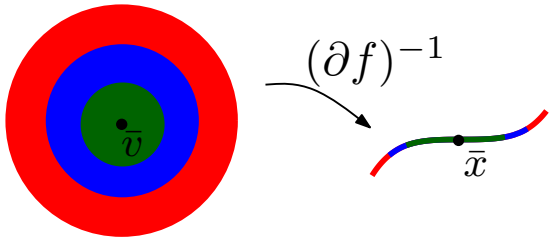

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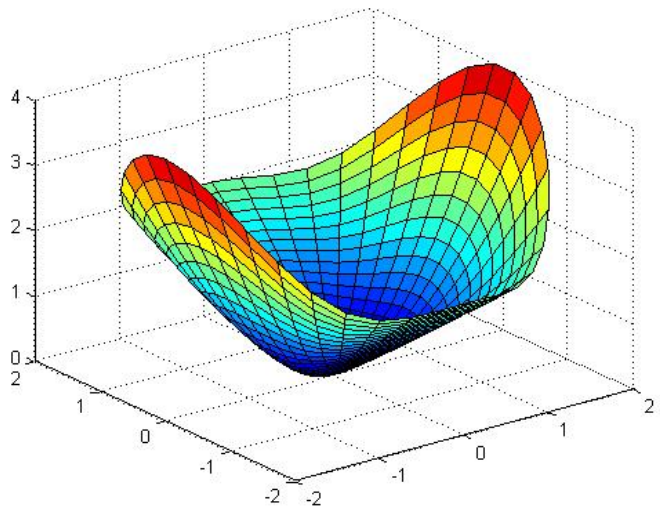

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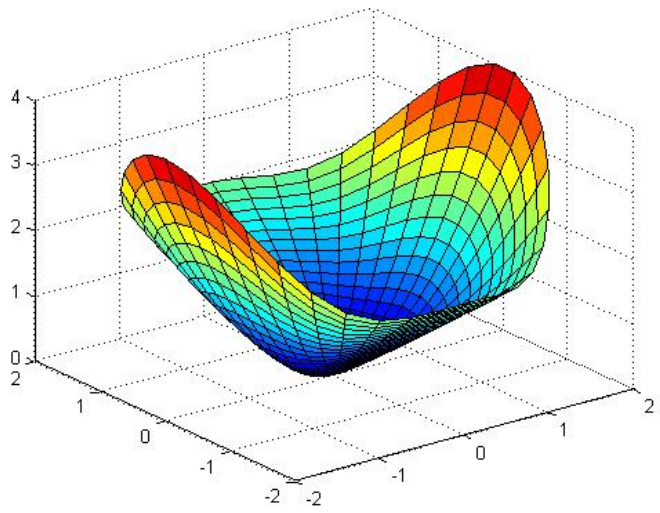

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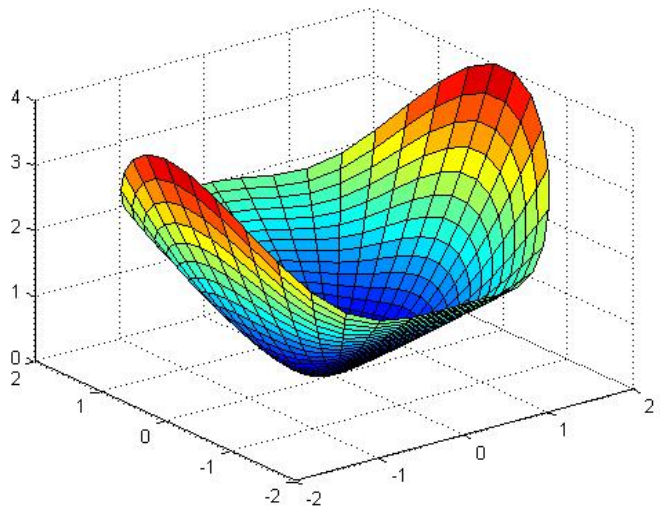

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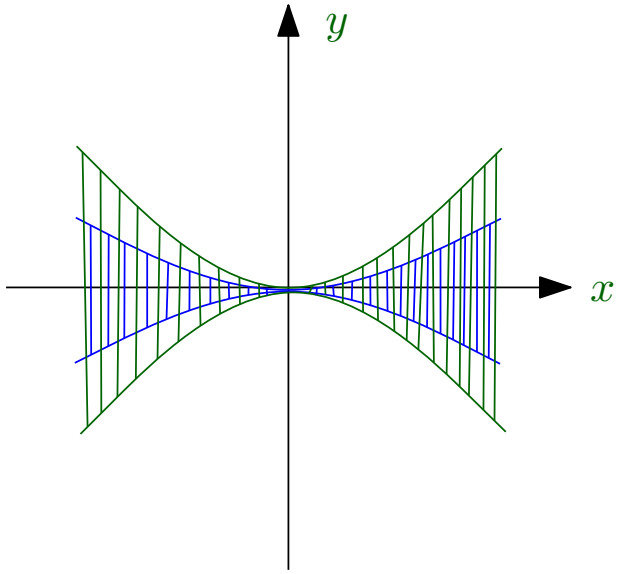

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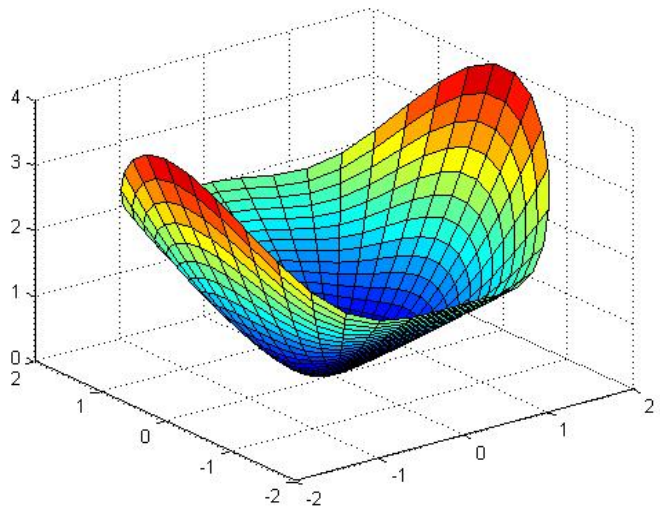

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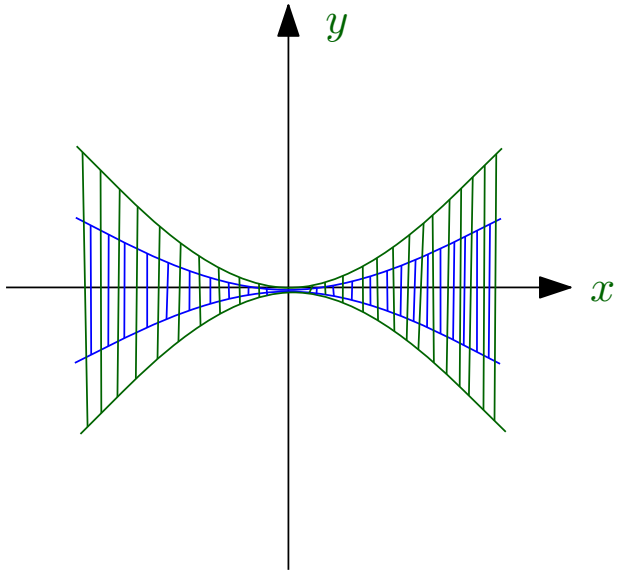

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Level sets of $|\nabla f|$ get "pinched".

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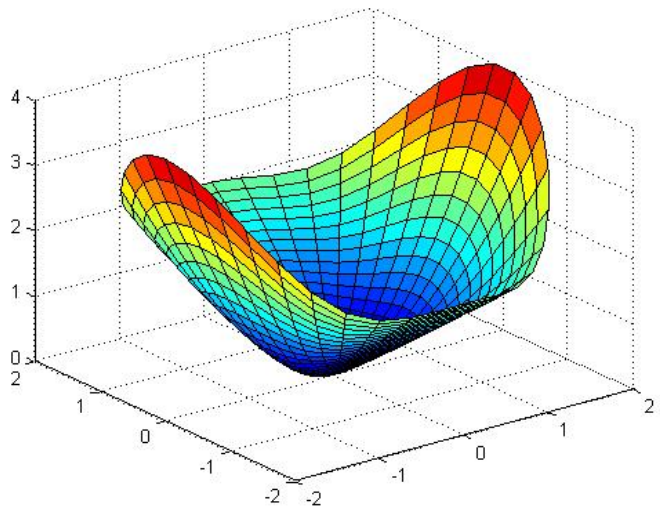

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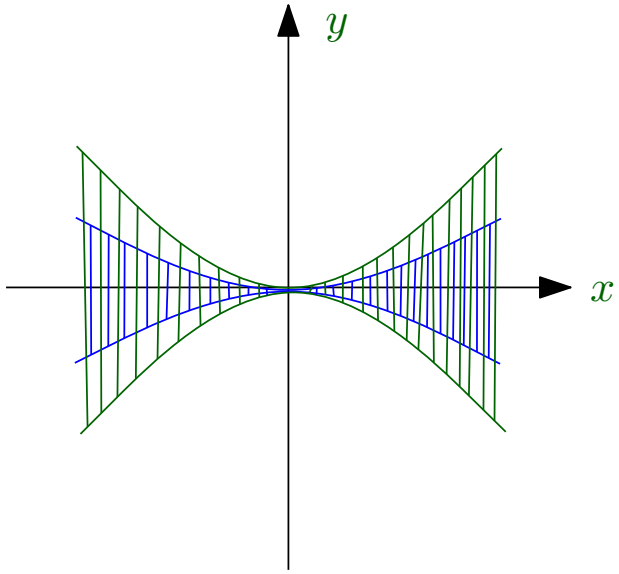

Figure: Level sets: $[|\nabla f| < \epsilon]$.

Level sets of $|\nabla f|$ get "pinched". No locally minimal identifiable set at $\bar{x} = (0,0)$ for $\bar{v} = (0,0)$.

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May use this to study nonpolyhedral variational inequalities!

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- One may try to exploit a nice identifiable set, if one exists; perhaps a $\mathbf{C}^2$-manifold.
- Use first order method and "track" M. Then make a guess and use Newton's method to accelerate (Survey by Sagastizábal '11 in SIAG/OPT Views and News).

Identifiable manifolds

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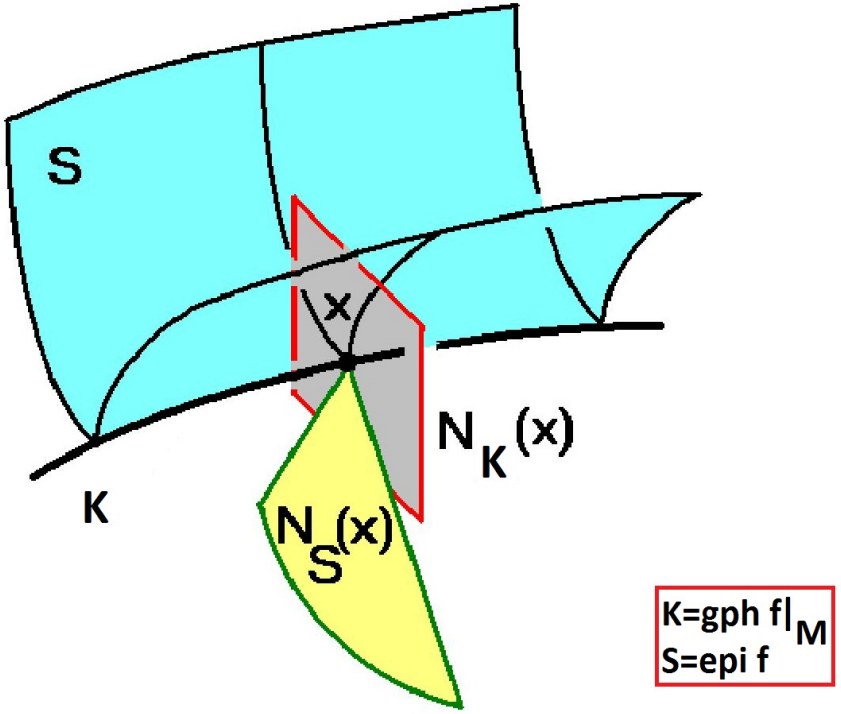

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- When an identifiable manifold exists, nonsmoothness is not intrinsic.
- So can reduce to the classical setting.

Generic Properties

History: Rockafellar-Spingarn '79, considered problems

$$P(v,u): \quad \min \quad f(x) - \langle v, x \rangle, \\ \text{s.t.} \quad g_i(x) \leq u_i, \text{ for all } i \in I := \{1, \ldots, m\},$$

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Theorem (Rockafellar-Spingarn '79)

- *For almost all v, active gradients are independent (there exists an active manifold).*
- *For almost all (v,u), the second order conditions hold at every minimizer:*

  Strict complementarity: multipliers are strictly positive and

  Quadratic growth: $f(\cdot) - \langle \bar{v}, \cdot \rangle$ grows quadratically on M near $\bar{x}$.

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Large class of sets for which the word typical has a canonical meaning.

Generic Properties

Theorem (D, Lewis)

Suppose $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ *is semi-algebraic. Consider the perturbed functions*

$$f_p(x) := f(x) + \theta(p, x),$$

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Then for a "typical" $p \in P$,

- *f_p has finitely many critical points x_p.*
- *f_p admits an identifiable manifold near each x_p for 0.*
- *every local minimizer x_p of f_p on the manifold is a strong local minimizer of f_p.*
- *Smooth dependence of critical points near x_p on the manifold.*

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- Dimension is a global property.

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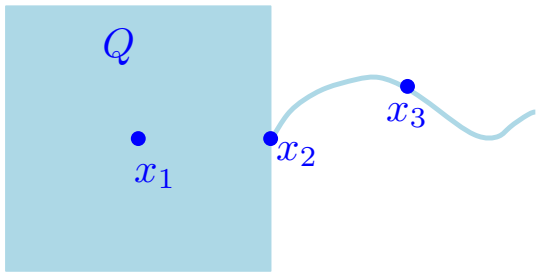

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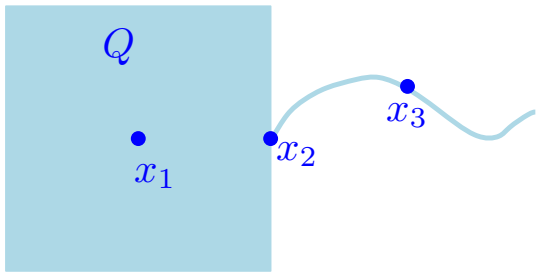

$\dim_Q(x_1) = \dim_Q(x_2) = 2$, $\dim_Q(x_3) = 1$

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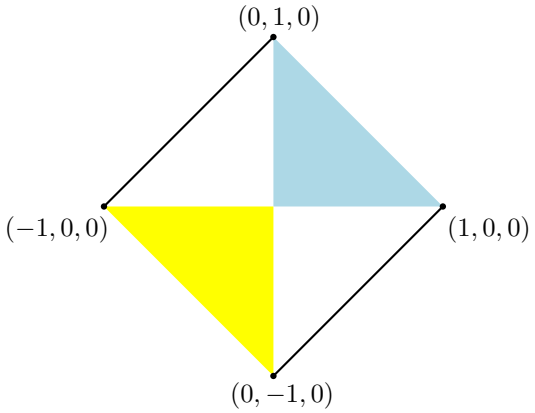

Figure: $\partial f(0,0,0)$

Work in progress

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Example

Semi-definite cone $\mathbf{S}^n_+$ stratifies into "identifiable manifolds"

$$S_k := \{X \in \mathbf{S}^n_+ : \operatorname{rank} X = k\}, \ \textit{for } k = 0, \ldots, n,$$

and is self-dual. Then have rigorous duality between S_k and S_{n-k}.

Summary

- Presented the intuitive notion of identifiable sets.
- Showed how identifiable sets capture the essence of many previously developed concepts (critical cones, Partial Smoothness, optimality conditions).
- Generic existence.

Thank you.