Variational analysis with substructure

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Goals

- Intuitive notion of identifiable sets.
- Existence, calculus, properties.
- Connection to critical cones (Generalized Reduction Lemma).
- Illustration: Spectral functions.
- Generic existence (semi-algebraic setting).
Definition (Generalized critical points)

$\bar{x}$ is a critical point of $f : \mathbb{R}^n \to \mathbb{R}$ if $0 \in \partial f(\bar{x})$. 
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\( \bar{x} \) is a critical point of \( f : \mathbb{R}^n \to \mathbb{R} \) if \( 0 \in \partial f(\bar{x}) \).

- For convex \( f \), critical points are global minimizers.
- If \( f \) is \( C^1 \)-smooth, criticality reduces to \( \nabla f(x) = 0 \).
Consider the **perturbed** functions

\[ f_v(x) = f(x) - \langle v, x \rangle. \]  

[For simplicity],

and suppose \( \bar{x} \) is **critical** for \( f_{\bar{v}} \), that is \( \bar{v} \in \partial f(\bar{x}) \).
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\textit{For simplicity}, and suppose \( \bar{x} \) is critical for \( f_v \), that is \( \bar{v} \in \partial f(\bar{x}) \).

\textbf{Sensitivity question}: How do solutions \( x_v \) of

\[ v \in \partial f(x), \]

vary, as we perturb \( v \) near \( \bar{v} \)?
Motivating Example

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Figure: $f(x, y) = x^2 + |y|, \quad M = \{(t, 0) : -1 < t < 1\}$
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- Only the restriction $f \big|_M$ matters!
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All perturbed solutions \(x_v\) of \(v \in \partial f(x)\) lie on \(M \implies M\) captures all the sensitivity information!
- Only the restriction \(f|_M\) matters!
- **Goal:** Look for small, well-behaved sets capturing only the essential information.
Finite identification

Consider the system

\[ \nu \in \partial f(x). \]
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**Definition (Identifiable sets)**
A set \( M \subset \mathbb{R}^n \) is identifiable at \( (\bar{x}, \bar{\nu}) \in \text{gph} \partial f \) if locally near \( \bar{x} \) have
\[ M = \pi_{\mathbb{R}^n}( (U \times V) \cap \text{gph} \partial f ), \]
for some neighbourhood \( U \times V \) of \( (\bar{x}, \bar{\nu}) \).
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**Example (Trivial example)**

**Example (Normal cone map)**

Let \( \partial f = N \mathcal{Q} \) for a cube \( \mathcal{Q} \subset \mathbb{R}^3 \).

In this case \( M = \bar{x} + K \mathcal{Q}(\bar{x}, \bar{v}) \).
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Why are identifiable sets interesting?
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Proposition (D, Lewis)

Suppose $M$ is an identifiable set at $(\bar{x}, 0) \in \text{gph} \partial f$.
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Why are identifiable sets interesting?

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Suppose $M$ is an identifiable set at $(\bar{x}, 0) \in \text{gph} \partial f$.

- $\bar{x}$ is a (strict) local minimizer of $f$ $\iff$ $\bar{x}$ is a (strict) local minimizer of $f$ on $M$.
- $f$ grows quadratically near $\bar{x}$ $\iff$ $f$ grows quadratically on $M$ near $\bar{x}$. 
Clearly all of $\text{dom } \partial f$ is identifiable at $(\bar{x}, \bar{v}) \in \text{gph } \partial f$ (not interesting). So...
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**Question**: What are the smallest possible identifiable sets?
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Definition
An identifiable set $M$ at $(\bar{x}, \bar{v}) \in \text{gph } \partial f$ is locally minimal if

$$M' \text{ identifiable at } (\bar{x}, \bar{v}) \implies M \subset M', \text{ locally near } \bar{x}.$$
Locally minimal identifiable sets exist for

- fully amenable functions: \( f(x) = g(F(x)) \) where
  1. \( F \) is \( C^2 \)-smooth,
  2. \( g \) is (convex) piecewise quadratic,
  3. qualification condition holds.

E.g. convex polyhedra, max-type functions, standard problems of nonlinear math programming.

A strong chain rule is available for composite functions \( f(x) = g(F(x)) \).
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The critical cone of a convex $Q$ at $\bar{x}$ for $\bar{v} \in N_Q(\bar{x})$ is

$$K_Q(\bar{x}, \bar{v}) := T_Q(\bar{x}) \cap \bar{v}^\perp.$$
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Important for analysing polyhedral variational inequalities

$$0 \in F(x, p) + N_Q(x),$$

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Important for analysing polyhedral variational inequalities

$$0 \in F(x, p) + N_Q(x),$$

where $Q$ is a convex polyhedron, because of
Proposition (Reduction Lemma due to Robinson)

If $Q$ is polyhedral, then

$$\text{gph } N_Q = \text{gph } N_{\bar{x} + K_Q(\bar{x}, \bar{v})} \text{ locally near } (\bar{x}, \bar{v}).$$
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Dimension Reduction

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Proposition (D, Lewis)

Let $M$ be a (prox-regular) identifiable set at $(\bar{x}, \bar{v}) \in gph \ N_Q(\bar{x})$. Then

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May use this to study nonpolyhedral variational inequalities!
Identifiable manifolds

$M$ is an identifiable manifold at $(\bar{x}, \bar{v}) \in \text{gph} \partial f$ if $M$ is identifiable, $M$ is a manifold, and $f \big|_M$ is smooth.

Proposition (D-Lewis) Identifiable manifolds $M \subset \text{dom} f$ are automatically locally minimal.

Identifiable manifolds provide a refinement of partly smooth manifolds introduced in Lewis'03. When an identifiable manifold exists, nonsmoothness is not intrinsic... So, can reduce analysis to the classical setting.
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Identifiable manifolds $M \subset \text{dom } f$ are **automatically locally minimal**.
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Lifts of identifiable manifolds

Consider $S^n := \{n \times n$ symmetric matrices$\}$ and the eigenvalue map

$$A \mapsto (\lambda_1(A), \ldots, \lambda_n(A)),$$

where

$$\lambda_1(A) \leq \ldots \leq \lambda_n(A).$$
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For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, invariant under permutation of coordinates, form the spectral function

$$f \circ \lambda : S^n \to \overline{\mathbb{R}}.$$
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Identifiable manifolds “lift”: (D, Lewis), (Daniilidis, Malick, Sendov)

$M$ identifiable manifold at $(\bar{x}, \bar{v}) \in \text{gph } \partial f$

$\implies \lambda^{-1}(M)$ identifiable manifold at $(\bar{X}, \bar{V}) \in \text{gph } \partial(f \circ \lambda)$. 
Generic Properties

History: Rockafellar-Spingarn ’79, considered problems

\[ P(v, u) : \min f(x) - \langle v, x \rangle, \]
\[ \text{s.t. } g_i(x) \leq u_i, \text{ for all } i \in I := \{1, \ldots, m\}, \]

for smooth \( f, g_i \).
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for smooth \( f, g_i \).

Theorem (Rockafellar-Spingarn ’79)

- For almost all \( (v, u) \), at every minimizer of \( P(v, u) \):
  
  - Active manifold: active gradients are independent
  - Strict complementarity: multipliers are strictly positive and
  - Quadratic growth: objective function grows quadratically.
**Generic Properties**

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**Theorem (D, Ioffe, Lewis)**

For semi-algebraic \( f : \mathbb{R}^n \to \mathbb{R} \), consider the perturbed functions

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f_{v}(x) := f(x) - \langle v, x \rangle,
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Then for a “typical” \( v \in \mathbb{R}^n \), at every minimizer \( x_v \) of \( f_v \),
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Then for a “typical” \( v \in \mathbb{R}^n \), at every minimizer \( x_v \) of \( f_v \),

- **Active manifold:** existence of an identifiable manifold
- **Strict complementarity:** \( 0 \in \text{ri} \, \partial f_v(x_v) \)
- **Quadratic growth:** \( f_v \) grows quadratically near \( x_v \).
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Semi-algebraic subdifferential graphs are not too big, not too small, but just right:
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Theorem (D, Ioffe, Lewis)

For lsc, semi-algebraic \( f : \mathbb{R}^n \to \bar{\mathbb{R}} \), we have

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\dim \text{gph } \partial f = n,
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Theorem (Ioffe)

For semi-algebraic $F : \mathbb{R}^n \Rightarrow \mathbb{R}^m$, for generic $v \in \mathbb{R}^m$, have

$$x \in F^{-1}(v) \implies F \text{ is metrically regular at } (x, v).$$
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Further, if $\dim \text{gph } F = n = m$, then strong metric regularity is typical.
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$$x \in F^{-1}(v) \implies F \text{ is metrically regular at } (x, v).$$

Further, if $\dim \text{gph} F = n = m$, then strong metric regularity is typical. Strong metric regularity of $\partial f$ (i.e. tilt-stability) is equivalent to a uniform quadratic growth condition (D, Lewis '12).
Presented the intuitive notion of identifiable sets.

Showed how identifiable sets capture the essence of previously developed concepts (dimension reduction, critical cones, optimality conditions).

Illustration: spectral functions.

Generic properties of semi-algebraic optimization problems.
Thank you.