The many faces of degeneracy in conic optimization

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Joint work with
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AFOSR YIP

Review of degeneracy

Primal-dual pair:

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Slater (D) often holds in applications, but Slater (P) may fail.

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Eg: Structured data

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More interesting examples later!

Detecting degeneracy

Exactly one holds (statement of alternative):

- Slater (P)
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such that the system

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(Renegar):

$$\text{distance to (P)-infeasibility} = \min_{y : \|y\| = 1} \max\{\|\lambda_-(\mathcal{A}^* y)\|, b^T y\}$$

Facial reduction

(P) feasible and y satisfies the auxiliary system $\Longrightarrow$

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Consider the region

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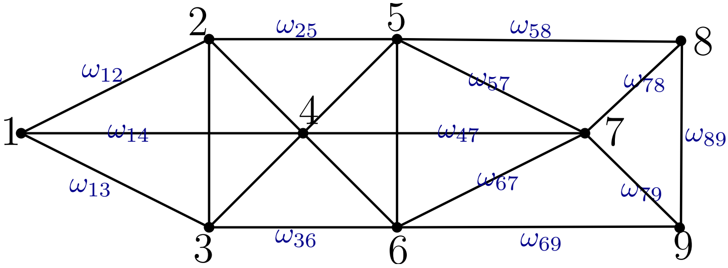
Eg. QAP, graph partitioning, second-lift of MAX-CUT.

Example: EDM completion

Problem: given a weighed graph $G = (V, E, \omega)$



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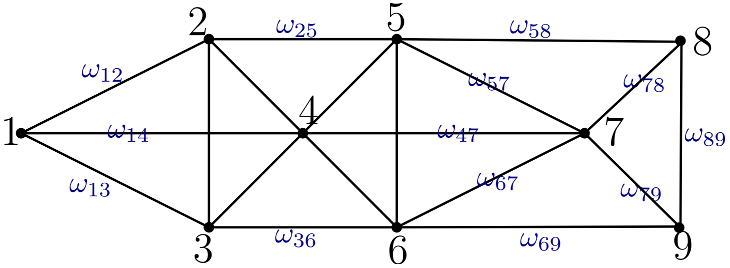

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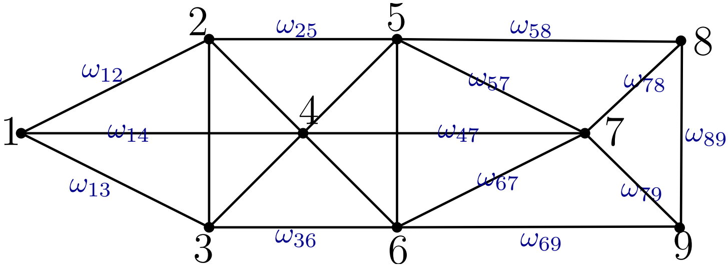

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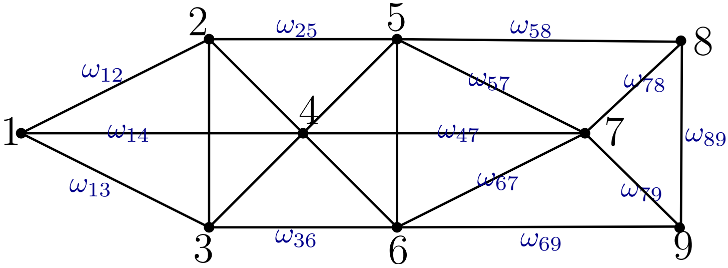

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Eg: Sensor network localization and molecular conformation

EDM completions

Natural substructures: cliques.

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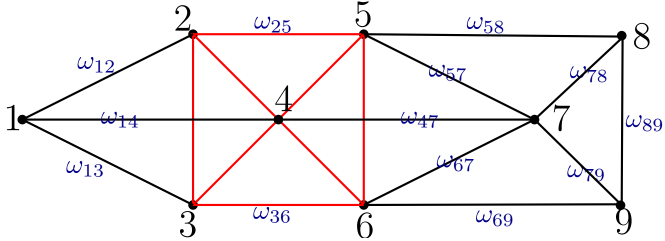

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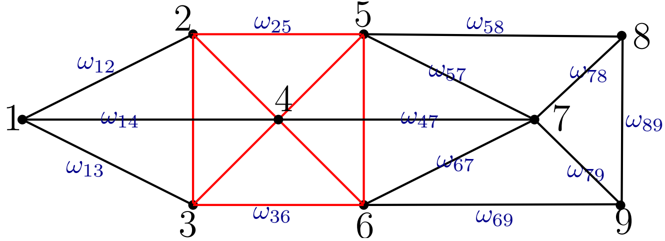

Idea: "Collapse" cliques

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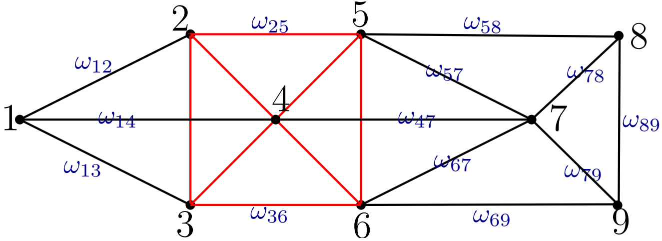

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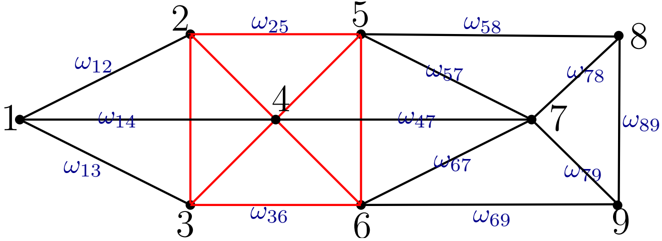

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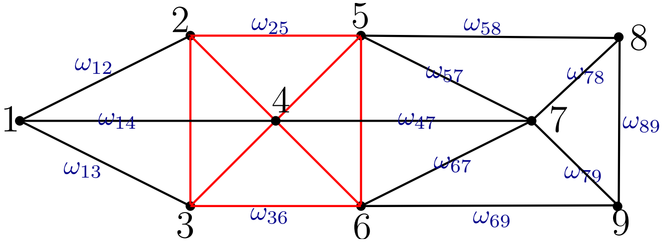

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Advertisement: see Krislock TD21 for more.

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5% noise, 6% density $(n = 1000, r = 2)$:

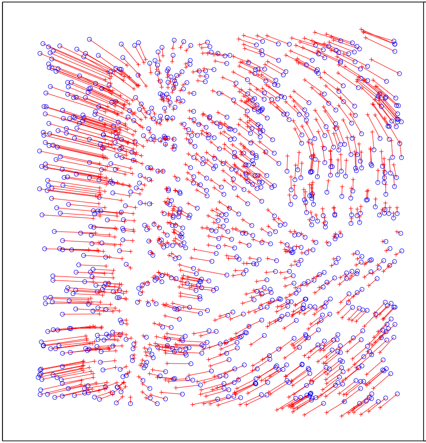

Figure: Before refinement

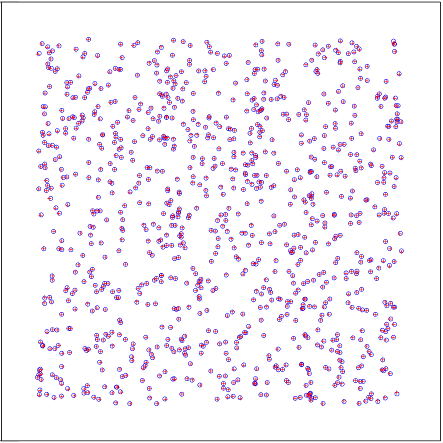

Figure: After refinement

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Intuition: $$\operatorname{tr}(X) = \frac{1}{2n} \sum_{i,j=1}^{n} \|p_i - p_j\|^2$$

Different idea for noisy EDM

Unfolding heuristic (Weinberger et al. '79):

$$\begin{aligned} \max \quad & \operatorname{tr}(X) \\ \text{s.t.} \quad & \sqrt{\sum_{ij \in E} |X_{ii} - 2X_{ij} + X_{jj} - \omega_{ij}|^2} \leq \sigma \\ & Xe = 0 \\ & X \succeq 0 \end{aligned}$$

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Flipped problem:

$$\begin{aligned} \psi(\tau) := \min \quad & \sqrt{\sum_{ij \in E} |X_{ii} - 2X_{ij} + X_{jj} - \omega_{ij}|^2} \\ \text{s.t.} \quad & \operatorname{tr} X = \tau \\ & Xe = 0 \\ & X \succeq 0. \end{aligned}$$

Approximate Newton

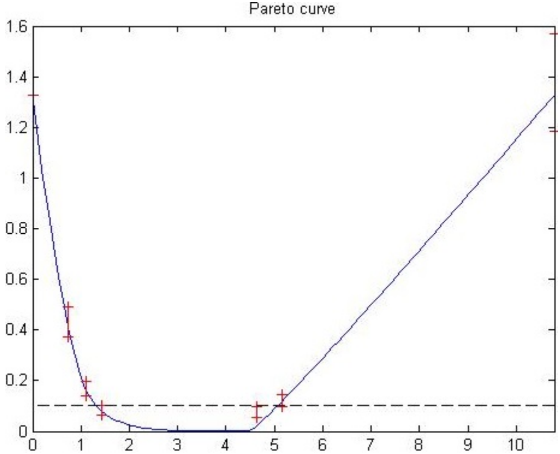

Figure: Graph of ψ

Strategy: approximate Newton method for finding

$$\boxed{\text{maximal } \tau \text{ with } \psi(\tau) \leq \sigma}.$$

Approximate evaluation of ψ with Frank-Wolfe algorithm.

Convergence guarantee: can obtain $X \succeq 0$ with

$$\operatorname{tr} X \geq \text{max-trace} \quad \text{and} \quad \text{residual} \ \leq \sigma + \epsilon$$

using

$$\mathcal{O}\left(\frac{\bar{\tau} \cdot \mathrm{Lip}^2}{\epsilon^2} \ln\left(\frac{(\tau_0 - \bar{\tau}) \cdot \psi_0'}{\epsilon}\right)\right) \quad \text{FW iterations.}$$

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Related “flippy strategies”: (van den Berg-Friedlander ’08, Harchaoui-Juditsky-Nemirovski ’13)

Max-trace vs Min-trace

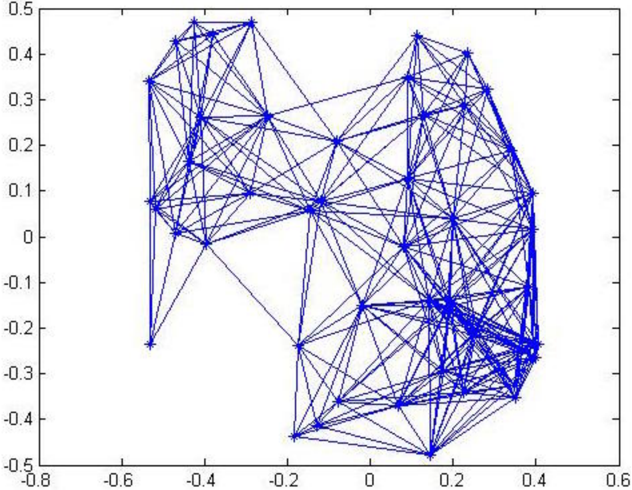
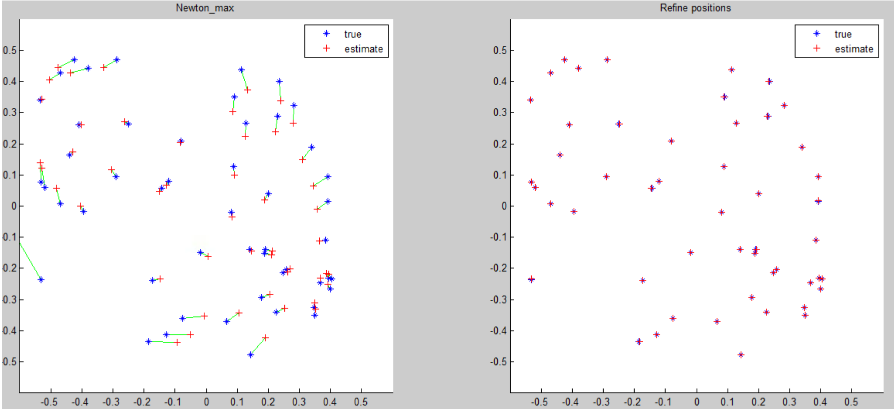


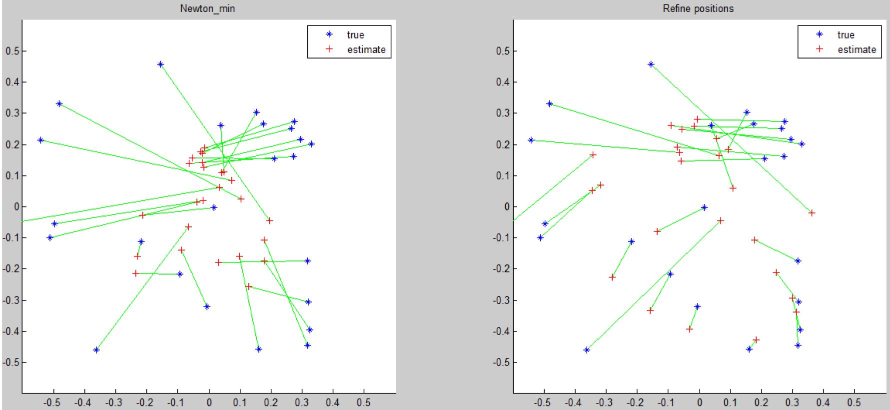
Figure: Sensor network

Max-trace



Newton_max
true
estimate
Refine positions
true
estimate
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5

Min-trace



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- Advertisement:
  Survey paper (with H. Wolkowicz) is forthcoming.

Thank you.