Minimization of convex compositions

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Joint work with
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AFOSR, NSF

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$$\min_{x} \; F(x) = g(x) + h(c(x))$$

Nonsmooth & Nonconvex minimization

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(Burke '85, Cartis-Gould-Toint '11, Fletcher '82, Lewis-Wright '15, Nesterov '06, Powell '84, Wright '90, Yuan '83, ...)

Examples

$$\boxed{\min_x \; g(x) + h(c(x))}$$

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- Additive composition:

$$\min_x \; g(x) + c(x)$$

- Least-norm misfit: $\min_{x \in Q} \; \|c(x)\|$

  - Robust Phase Retrieval: $\min_x \sum_{i=1}^{m} |(a_i^T x)^2 - b_i|$
  - Nonneg. Factorization: $\min_{X,Y \geq 0} \; \|XY^T - D\|$

Prox-linear algorithm

$$\boxed{\min_{x} \; F(x) = g(x) + h\big(c(x)\big)}$$

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Local Model:

$$F_x(y) := g(y) + h\Big(c(x) + \nabla c(x)(y - x)\Big)$$

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Accuracy: $$|F_x(y) - F(y)| \leq \frac{\beta}{2}\|y - x\|^2 \qquad \forall x, y$$

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Prox-linear method (Burke, Fletcher, Nesterov, Powell, ...):

$$x^+ = \operatorname*{argmin}_y \; \left\{F_x(y) + \frac{\beta}{2}\|y - x\|^2\right\}$$

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Big assumption: x^+ is computable (for now)

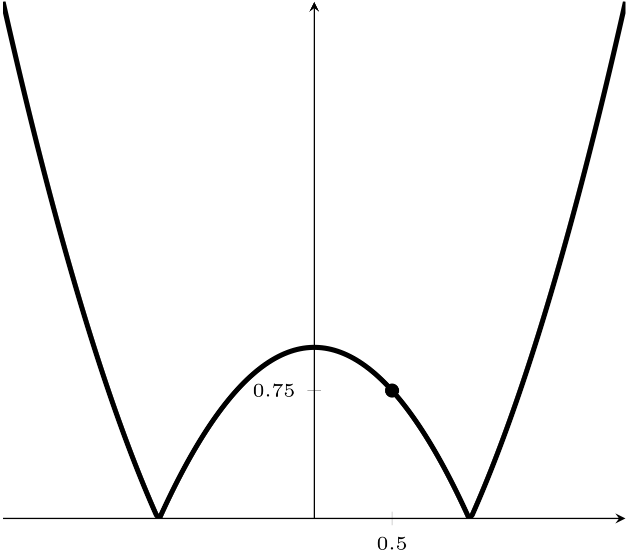
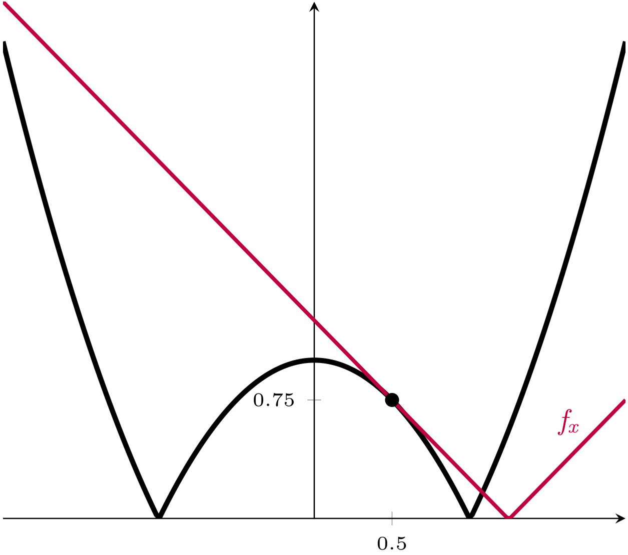


Figure: $f(x) = |x^2 - 1|$



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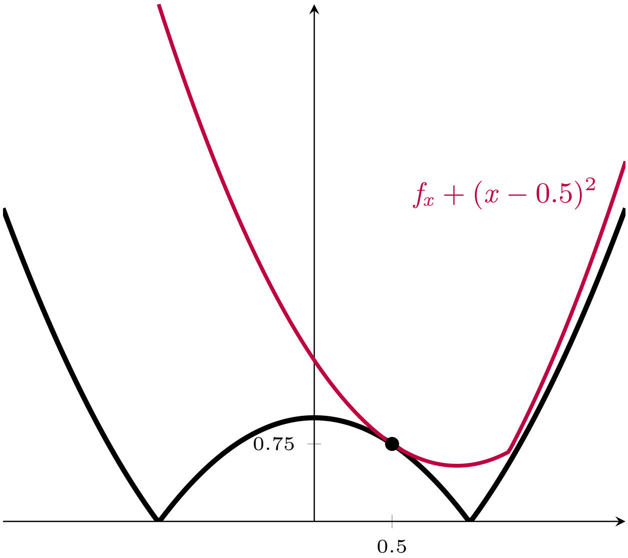

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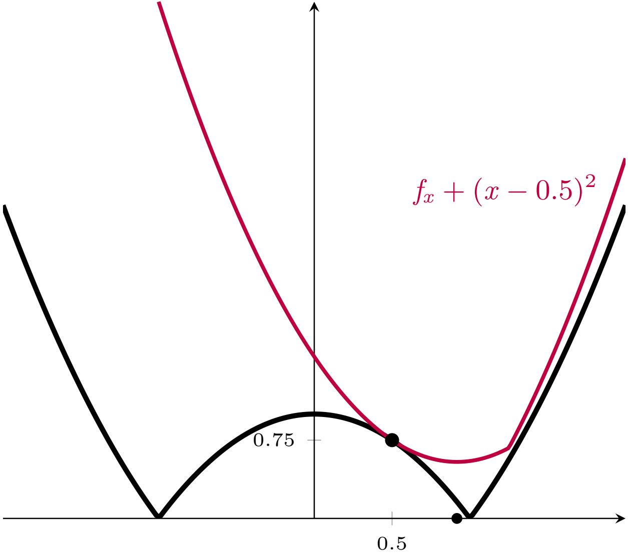

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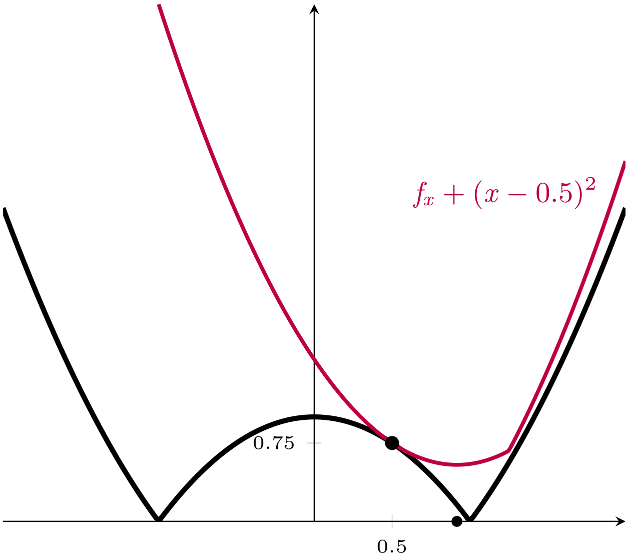

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- No finite termination

Sublinear rate

Prox-gradient:

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Philosophy (Nesterov '13): $x^+ = x - \frac{1}{\beta}\mathcal{G}(x)$ and so

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$$\|\mathcal{G}(x_k)\| < \epsilon \quad \text{after} \quad \mathcal{O}\left(\frac{\beta}{\varepsilon^2} \cdot (F(x_0) - F^*)\right) \text{ iterations}$$

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- d, m are large $\implies$ x^+ inexactly computable

First-order complexity

First-order complexity of a method:

Max # of

- $c(x), \nabla c(x) v$, $\nabla c(x)^T v$,
- $\operatorname{prox}_{th}(x), \operatorname{prox}_{tg}(x)$

a method evaluates to produce $\|\mathcal{G}(x)\| \leq \varepsilon$.

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Thm: (D-Paquette '16) Define

$$\|\nabla c\| := \max_{x \in \text{dom } g} \|\nabla c(x)\|.$$

There exists a method with complexity

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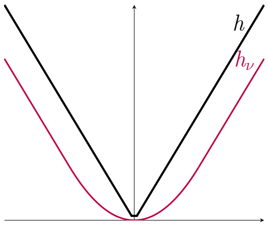
Strategy:

- smoothing of h,
- inexact prox-linear method,
- fast-gradient subsolve.

Smoothing

Moreau envelope: $$h_\nu(x) := \min_y \left\{h(y) + \frac{1}{2\nu}\|y - x\|^2\right\}.$$



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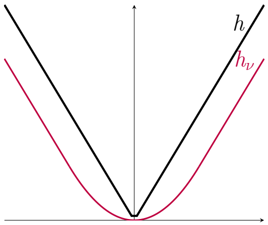

Lemma: For $\nu > 0$, consider

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Then

$$\|\mathcal{G}(x)\| \leq \|\mathcal{G}^\nu(x)\| + \sqrt{\frac{\beta\nu}{2}}.$$

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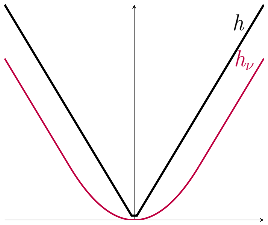

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Therefore set $\frac{\varepsilon}{2} = \sqrt{\frac{\beta\nu}{2}} \quad \Longrightarrow \quad \nu = \frac{\varepsilon^2}{2\beta}$.

Inexact prox-linear

Input: A point $x_0 \in \operatorname{dom} g$, a method $\mathcal{M}$
Step k: $(k \geq 1)$
Consider: $\min_y \; \varphi(y) := F_{x_k}(y) + \frac{\beta}{2}\|y - x_k\|^2.$

Suppose $\mathcal{M}$ is linearly convergent from $y_0 = x_k$:

$$\varphi(y_i) - \varphi(y^*) \leq \gamma(1-\tau)^i \|y_0 - y^*\|^2 \qquad \forall i = 1, 2 \ldots$$

Set $x_{k+1} = y_T$ where $T := \left\lceil \frac{1}{\tau} \log(4\gamma/\beta) \right\rceil$.

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$\|\mathcal{G}(x_k)\| < \epsilon$ after $\mathcal{O}\left(\frac{\beta}{\varepsilon^2} \cdot (F(x_0) - F^*)\right)$ outer iterations.

$\Longrightarrow$ smoothing+prox-linear+fast-gradient gives complexity

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Strategy:

- smoothing of h_i,
- inexact prox-linear method,
- accelerated incremental subsolves:
  Lin-Mairal-Harchaui '15, Shalev-Shwartz '15,
  Frostig-Ge-Kakade-Sidford '15, Lan '15, Allen-Zhu '16

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Additive composite $F = c + g$ setting:

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$$\Longrightarrow \mathrm{dist}\Big((x, 0); \mathrm{gph}\, \partial F\Big) \leq \|\mathcal{G}(x)\|.$$

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Setting: $S := \{x : 0 \in \partial F(x)\}$, $\bar{x} \in S$, $\bar{x} \in \operatorname{int} \mathcal{X}$

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2) Tilt-stability: (Poliquin-Rockafellar '98)

$$v \mapsto \operatorname*{argmin}_{x \in \mathcal{X}} \{F(x) - \langle v, x \rangle\} \qquad \text{is } \tfrac{1}{\alpha}\text{-Lipschitz near } \bar{v} = 0$$

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Local rates

Regularity	**Convergence Guarantee**
Error-bound	$\frac{F(x_{k+1})-F^*}{F(x_k)-F^*} \leq 1 - \left(\frac{\alpha}{\beta}\right)^2$
Tilt-stability	$\frac{F(x_{k+1})-F^*}{F(x_k)-F^*} \leq 1 - \frac{\alpha}{\beta}$
Sharpness	$\|x_{k+1} - x^*\| \leq \mathcal{O}(\|x_k - x^*\|^2)$

(Nesterov '06, D-Lewis '15)

Recent related work

"Accelerated" prox-linear method (D.-Paquette '16):

$$\min_{j=1,\ldots,k} \|\mathcal{G}(x_i)\|^2 \leq (\beta M)^2 \cdot \mathcal{O}\left(k^{-3} + \rho \cdot k^{-1}\right),$$

assuming $M = \operatorname{diam}(\operatorname{dom} g)$ finite and $\rho \in (0,1)$ satisfies

$$F_x(y) \leq F(y) + \rho \cdot \tfrac{\beta}{2}\|y - x\|^2 \qquad \forall x, y$$

Sampling methods (Duchi-Ruan '16): Almost sure conv. on

$$\min_x \; g(x) + \int_{\mathcal{S}} h(c(x,s), s)\, dP(s).$$

Inner-outer subgradient methods w/ rates (Davis-Grimmer '17)

Robust Phase Retrieval (Duchi-Ruan '17):
Quadratic convergence w.h.p on $\quad \min_x \; \frac{1}{m}\sum_{i=1}^{m} |(a_i^T x)^2 - b_i|$
[uses Eldar-Mendelson '12]

Summary

"Efficiency of minimizing compositions of convex functions and smooth maps", D-Paquette, 2016, arXiv:1605.00125.

"Error bounds, quadratic growth, and linear convergence of proximal methods", D-Lewis, Math. Oper. Res. 2017.

- prox-linear method
- first-order complexity
- finite-sum problems
- termination conditions
- local linear/quadratic rates

Thank you!