(1) It was YOUR suggestion that after an "error", all computed expressions should lie in

$$A = R[\frac{\epsilon}{y_i}]$$

where the y_i vary over a suitable set of computed expressions. In the Dodgson situation that would say that y_i are from the set:

$$a_{2,2}, b_{1,2}, b_{2,1}, C_{1,1}$$

.

This agrees with previous calculations. That is,with suitable starting data, each of these y_i does occur as the dominant loss of precision, but no other y_i need be considered.

I'm not quite sure what sparked this, but you are completely right!

Namely, the version of Robbins for cluster algebras that I have been tending to give when asked for the most general possible statement is flawed: it doesn't reproduce to what we think Robbins' is, because it allows for too many denominators. So what I've been proposing is easier (and therefore even more likely to be true), but isn't the "right" statement. Notice that Robbins didn't have to deal with this subtle point because he just worked over a DVR (rather than algebraically) *and* he fixed the precision and didn't care about anything other than the "final" error. Perhaps this issue does illustrate the merits of the algebraic point of view (e.g., in some cases it predicts, correctly we hope, a smaller error than Robbins would).

After thinking about this overnight I'm convinced that the correct conjecture is that the denominators are the generalization of exactly what you suggest. *Namely* the computed value should be a polynomial in

$$\frac{\epsilon}{X}$$

where ϵ is one of the errors, and X is *either* the divisor at the exchange where ϵ is an error *or* one of the variables in the exchange polynomial that is in the monomial multiplied by (1+eps).

However, this is capable of serious testing, modulo some heuristic assumptions. Namely, to assess which errors need to be considered, it seems highly likely that we can restrict to the case of first order errors $\frac{\epsilon}{X}$ that occur in *some* DVR instance. So this is quite easy to test rigorously, and I will try to do this soon, or at least before you get here.

(2) A "weak" conjecture would say that the valuations behave as though this were the case. I am trying to see if the cluster algebra approach can be adapted to handle valuations.

Because of my work with Kiran, I'm locked in to using the term weak to mean that the valuation-theoretic statement that has a multiplier m ¿ 1 to the Robbins final precision p-c.

The DVR version should apply just fine to cluster algebras.

(2') You may have noticed that in my previous e-mail, I implied that I can show a "quite weak" version, as follows:

The loss of precision is at most the sum of the valuations of the four denominators above.

This is accomplished by using an auxilliary indeterminate δ, setting the error as $b_{1,1}b_{2,2}b_{1,2}b_{2,1}C_{1,1}\delta$. All computations are in the UFD $A[\delta]$. Finally, substitute

$$\delta = \frac{\epsilon}{a_{2,2}b_{1,2}b_{2,1}C_{1,1}}$$

I think that Kiran and Roe both have versions of this (or, in the case of Roe, his work seems to imply such a version if his idea is applied to Dodgson). Roe's is actually even weaker than yours, and Kiran's would be slightly better (in a natural way that I think that you could derive in the base case using your method).

(3) You say:

b) Any notation for Dodgson is "unavoidably heavy and unwieldy, and tends to obscure key points."

You seem to want a cluster algebra approach that depends only on binomial exchange relations so that the basis step of an induction proof goes through.

BUT, the difficulty seems to be lack of UFD property. I am trying out the idea of using Dodgson clusters (all UFD's) with a "frozen" indeterminate δ as above.

By basis step there I only meant the basis step of the Laurent proof (minus the GCD assertions). I think that the basis step of Robbins should work the same for clusters as your Dodgson proof. The lack of the UFD property with epsilon's is a problem for the induction step.

You said: (5) "I was about to type in the Dodgson-to-cluster interpretation, but not sure whether you want to see that, now."

See below.

(5') I agree that any cluster algebra approach makes use of some sort of Caterpillar lemma. In Lam's case, it is hidden in the assumptions.

There is also the proof in Cluster III, but I now think (along with Kiran) that this proof is, basically, "homotopic" to the caterpillar proof in the case of binomial exchange relations.

(6) I am trying to make precise the Somos-5, Somos-6, and Nomos-6 cases. What is the status of them? Can you explain what goes wrong with Nomos-7.

I don't think that Robbins, as we understand it, is true unless the exchange relations are binomial. I have the fantasy that once Robbins is understood, a weak Robbins will be true with a multiplier *only* determined by what happens in the basic 3-step case.

Somos-5 is "nice" and therefore the idea of the Somos-4 proof applies, and the full algebraic Robbins is true.

Somos-6, by which I mean (in notation that we used earlier)

$$06 = 15 + 24 + 3^2$$

does not have binomial exchange relations. Laurent is true, and, empirically, a weak form of Robbins is true (if I recall, which I well might not, Kiran might be able to actually prove the DVR conjecture for some multiplier).

Nomos-6 is Laurent and, empirically, satisfies strong Robbins (and I think Kiran's idea proves weak-Robbins for some multiplier).

I'm not sure what Nomos-7 means. Somos-7 would be

$$07 = 16 + 25 + 34.$$

1. (Geometric) CLUSTER ALGEBRAS. Fix $n > 1$, and a rational function field $F = Q(x_1, ..., x_n)$ generated by n indeterminates. A node is a pair (u, T) where u is an n-tuple of elements of F (that is a transcendence base for F over Q), and T is an n-tuple of binomial polynomials in n variables (the sum of two relatively prime monomials) in which the i-th entry T_i does not depend on x_i. Such as set T is skew-symmetric if it is defined, in the usual cluster algebra way, by a skew-symmetric matrix, i.e.,

$$T_i(X_1, ..., X_n) = m^+ + m^-$$

where m^+ (resp m^-) is the product of all X_j^a (resp X_j^-a) such thatthe j-th element of the i-th row of the matrix is $a > 0$ (resp $a < 0$). Assume that we have a skew-symmetric node (u, T) in which u is $(x_1, \ldots, x_n)$. This can be propagated to the full n-regular tree in which each node has n edges with each edge labeled by a unique integer $i \in [n]$. Traversing the graph from a node (u, T) to a node (v, S) along an edge labeled i (i.e., a mutation in direction i) is done in the usual cluster algebra way: all of the variables are unchanged except for the i-th, where the new variable v_i is defined by

$$(*) \qquad u_i v_i = T_i(u_1, \ldots, u_n).$$

All exchange polynomials are changed *except* the i-th, essentially by change of variables. More precisely, S_j is obtained from T_j by the following steps:

1) replace X_i in $T_j(X_1, \ldots, X_n)$ by $T_i(X_1, \ldots, X_n/X_i)$

(**) hskip1.5 2) set $X_j = 0$

3) factor out a Laurent monomial to get a coprime binomial.

Consider a path from the initial node (x, T) to some other node (u, S). We can now consider the caterpillar Lemma after attaching n-2 nodes to each of the internal vertices on the path. Robbins, either DVR or algebraic, can now be formulated.

2. DODGSON CONDENSATION. How does the above square with Dodgson condensation?

Fix n. Let $A = [n+1]^2 \times [n]^2$ be the set of points that we visualize as a $n+1$ by $n+1$ grid of dots, with an n by n grid interspersed in the center of the squares, and one unit upwards. (i.e., the first two layers of Dodgson, where we are going to start with an all-1 matrix of size $n+1$ by $n+1$)

Let $N = (n+1)^2 + n^2$ be the cardinality of A, and let $F = Q(x_{1,1}, ..., y_{n,n})$ be the rational function field generated by indeterminates corresponding to elements of A and indexed by positions (in, say, an x-layer and a y-layer).

A node of the cluster algebra is a pair (u, T) consisting of an N-tuple of elements in F, that is a transcendence base, and a set of N exchange polynomials, which all are of the Dodgson condensation form. The *only* thing that we care about is the rather long caterpillar path defined by:

Mutate, in *any* order (!), the $(n-1)^2$ internal vertices of the lowest level to being in the third level.

Mutate, again in *any* order, the $(n-2)^2$ internal vertices in the n^2 level 2 to level 4.

And so on, ending with a mutation to the 1×1 determinant at the top.

You might ask what happens when we stray off such a path, i.e., mutate things in different layers, out of turn. Well, that corresponds to straying off the spine in the Somos sequence. It's still in the cluster algebra, Robbins probably still holds, but in each case it's less interesting given the original motivations. Notice that "substitution of variables commutes" (actually, given step 2 in (**) this has to be checked), so the order of mutations in each layer doesn't matter. This is one of the confusing points of F-Z, where they don't go to great length to make that clear.

In any event, Robbins for Dodgson is subsumed by Robbins for cluster.

I think that there are two technical errors in the above: (A) the 2x2 determinant in Dodgson has a minus sign whereas the vanilla cluster algebra only has plus signs, and (B) if you do stray off the Dodgson path in this cluster algebra the exchange polynomials are *not* of the Dodgson type. But these can be remedied easily (for (A), allow ± 1 coefficients in the exchange polynomials, or (better) use frozen variables, and for (B), just don't stray off any of the Dodgson paths or (better) just work in the whole cluster algebra.