04/17/14 cluster

Let $\pi = \{x_{i,j}\}$, be the starting plane of indeterminates, and let $R = Z[x_{i,j}]$ be the polynomial ring with these indeterminates. For each $n \geq 1$, let $x_{i,j}(n)$ be the $n \times n$ minor with $x_{i,j}$ in the upper left corner. The $x_{i,j}(n)$ satisfy the Dodgson recursion: For $n \geq 3$,

$$x_{i+1,j+1}(n-2) \cdot x_{i,j}(n) = x_{i,j}(n-1) \cdot x_{i+1,j+1}(n-1) - x_{i,j+1}(n-1) \cdot x_{i+1,j}(n-1)$$

The position of $x_{i,j}(n)$ will be designated (i,j,n). **Claim:** $E_{0,0}$ is in A

Proof: Let $Y = D_{0,0}D_{1,1} - D_{0,1}D_{1,0}$. Thus $E_{0,0} = \dfrac{Y}{C_{1,1}}$. Multiply Y by $b_{1,1}$, to get

$$\begin{aligned} b_{1,1}Y &= D_{1,1}\big(c_{0,0}C_{1,1} - c_{0,1}c_{1,0}\big) - b_{1,1}D_{0,1}D_{1,0} \\ &= C_{1,1}X_1 - U \end{aligned}$$

where $X_1 = D_{1,1}c_{0,0}$, and $U = c_{0,1}c_{1,0}D_{1,1} + b_{1,1}D_{0,1}D_{1,0}$.

$$\begin{aligned} b_{2,1}U &= b_{2,1}c_{0,1}c_{1,0}D_{1,1} + b_{2,1}b_{1,1}D_{0,1}D_{1,0} \\ &= b_{2,1}c_{0,1}c_{1,0}D_{1,1} + b_{1,1}D_{0,1}\big(c_{1,0}c_{2,1} - C_{1,1}c_{2,0}\big) \\ &= c_{1,0}V - C_{1,1}X_2 \end{aligned}$$

where $\quad X_2 = b_{1,1}D_{0,1}c_{2,0} \quad$ and $V = b_{2,1}D_{1,1}c_{0,1} + b_{1,1}D_{0,1}c_{2,1}$

$$\begin{aligned} b_{2,2}V &= b_{2,2}b_{2,1}D_{1,1}c_{0,1} + b_{2,2}b_{1,1}D_{0,1}c_{2,1} \\ &= b_{2,1}c_{0,1}\big(C_{1,1}c_{2,2} - c_{1,2}c_{2,1}\big) + b_{2,2}b_{1,1}D_{0,1}c_{2,1} \\ &= C_{1,1}X_3 + c_{2,1}W \end{aligned}$$

where $X_3 = b_{2,1}c_{0,1}c_{2,2}$ and $W = -b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}D_{0,1}$

$$\begin{aligned} b_{1,2}W &= -b_{1,2}b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}b_{1,2}D_{0,1} \\ &= -b_{1,2}b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}\big(c_{0,1}c_{1,2} - c_{0,2}C_{1,1})\big) \\ &= -b_{1,1}b_{2,2}c_{0,2}C_{1,1} + c_{0,1}c_{1,2}(-b_{1,2}b_{2,1} + b_{1,1}b_{2,2}) \\ &= -b_{1,1}b_{2,2}c_{0,2}C_{1,1} + c_{0,1}c_{1,2}(a_{2,2}C_{1,1}) \\ &= X_4C_{1,1} \end{aligned}$$

where

$$\begin{aligned} X_4 &= a_{2,2}c_{0,1}c_{1,2} - b_{1,1}b_{2,2}c_{0,2} \\ &= a_{2,2}c_{0,1}c_{1,2} - a_{2,2}c_{1,1}c_{0,2} - b_{1,2}b_{2,1}c_{0,2} \\ &= a_{2,2}b_{1,2}d_{0,1} - b_{1,2}b_{2,1}c_{0,2} \end{aligned}$$

$$\begin{aligned} FY &= b_{1,2}b_{2,2}b_{2,1}C_{1,1}X_1 \\ &- b_{1,2}b_{2,2}C_{1,1}X_2 \\ &- b_{1,2}c_{1,0}C_{1,1}X_3 \\ &- c_{1,0}c_{2,1}C_{1,1}X_4 \\ \\ &= C_{1,1}\big(b_{1,2}b_{2,2}b_{2,1}D_{1,1}c_{0,0} \\ &- b_{1,2}b_{2,2}b_{1,1}c_{2,0}D_{0,1} \\ &- b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} \\ &- c_{1,0}c_{2,1}X_4\big) \\ \\ &= C_{1,1}\big(b_{1,2}b_{2,2}b_{2,1}c_{0,0}\big(d_{1,1}\big) \\ &- b_{1,2}b_{2,2}b_{1,1}c_{2,0}\big(d_{0,1}\big) \\ &- b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} + c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}\big) \end{aligned}$$

A calculation (reversing the steps above), shows that

$$\begin{aligned} FY &= C_{1,1}\big(b_{1,2}b_{2,2}b_{2,1}c_{0,0}d_{1,1} - b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} \\ &- b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} + c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}\big) \\ &= C_{1,1}x \end{aligned}$$

where

$$\begin{aligned} x &= b_{1,2}b_{2,2}b_{2,1}c_{0,0}d_{1,1} - b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} - b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} \\ &+ c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2} \end{aligned}$$

(1) If we set ϵ to 0, then Y becomes $y = d_{0,0}d_{1,1} - d_{0,1}d_{1,0}$, and we have

$$b_{1,1}b_{2,2}b_{1,2}b_{2,1}y = c_{1,1}x$$

Each of these expressions $(b_{1,1}b_{2,2}b_{1,2}b_{2,1}, y, c_{1,1}, x)$ is a polynomial in the original ring $Z[x_{i,j}]$, and $c_{1,1}$ is indecomposable. Therefore

$$\frac{x}{F} = \frac{y}{c_{1,1}} = e_{0,0}$$

(2) The expression for Z must now be divided by $C_{1,1}$ The first and second terms are easy:

$$\begin{aligned} b_{1,2}b_{2,1}c_{0,0}c_{2,2}\delta &= c_{0,0}c_{2,2}\frac{\epsilon}{a_{2,2}} \\ b_{1,2}b_{2,1}c_{2,0}c_{2,0}\delta &= c_{2,0}c_{0,2}\frac{\epsilon}{a_{2,2}} \end{aligned}$$

(3) Let $P = \Big(c_{1,0}c_{2,1}c_{0,1}c_{1,2} - C_{1,1}c_{1,1}c_{2,0}c_{0,2}\Big)a_{2,2}\delta$. Using

$$c_{1,0}c_{2,1} = b_{2,1}D_{1,0} + c_{2,0}C_{1,1}$$

$$c_{0,1}c_{1,2} = b_{1,2}D_{0,1} + c_{0,2}C_{1,1},$$

we have

$$\begin{aligned} P &= \Big((b_{2,1}D_{1,0} + c_{2,0}C_{1,1}) \cdot (b_{1,2}D_{0,1} + c_{0,2}C_{1,1}) - C_{1,1}c_{1,1}c_{2,0}c_{0,2}\Big)a_{2,2}\delta \\ &= b_{2,1}b_{1,2}D_{1,0}D_{0,1}a_{2,2}\delta \\ &+ b_{2,1}D_{1,0}c_{0,2}C_{1,1}a_{2,2}\delta \\ &+ c_{2,0}b_{1,2}D_{0,1}C_{1,1}a_{2,2}\delta \\ &+ \Big(c_{2,0}C_{1,1}c_{0,2}C_{1,1} - C_{1,1}c_{1,1}c_{2,0}c_{0,2}\Big)a_{2,2}\delta \\ &= D_{1,0}D_{0,1}\epsilon \\ &+ D_{1,0}c_{0,2}C_{1,1}\frac{\epsilon}{b_{1,2}} \\ &+ c_{2,0}D_{0,1}C_{1,1}\frac{\epsilon}{b_{2,1}} \\ &+ C_{1,1}c_{2,0}c_{0,2} \cdot (C_{1,1} - c_{1,1}) \cdot \frac{\epsilon}{b_{1,2}b_{2,1}} \end{aligned}$$

(3a) After dividing by $C_{1,1}$, the first term becomes $D_{1,0}D_{0,1}\frac{\epsilon}{C_{1,1}}$, which is in A. (After expanding the definitions of $D_{1,0}, D_{0,1}$, this will involve some quadratic and cubic terms in ϵ.)

(3b) After dividing by $C_{1,1}$, the second term becomes $D_{1,0}c_{0,2}\frac{\epsilon}{b_{1,2}}$, which is in A.

(3c) After dividing by $C_{1,1}$, the third term becomes $c_{2,0}D_{0,1}\frac{\epsilon}{b_{2,1}}$, which is in A.

(3d) After dividing by $C_{1,1}$, and using $C_{1,1} - c_{1,1} = \frac{b_{1,1}b_{2,2}\epsilon}{a_{2,2}}$, the fourth term becomes

$$c_{2,0}c_{0,2} \cdot \frac{b_{1,1}b_{2,2}\epsilon}{a_{2,2}} \cdot \frac{\epsilon}{b_{1,2}b_{2,1}}.$$

Using the identity:

$$\frac{b_{1,1}b_{2,2}\epsilon}{b_{1,2}a_{2,2}} = c_{1,1}\frac{\epsilon}{b_{1,2}} + b_{2,1}\frac{\epsilon}{a_{2,2}}$$

this becomes

$$c_{2,0}c_{0,2} \cdot \left(c_{1,1}\frac{\epsilon}{b_{1,2}} + b_{2,1}\frac{\epsilon}{a_{2,2}}\right)\frac{\epsilon}{b_{2,1}}$$

which is in A. (Some terms quadratic in ϵ appear.) The net result is that $\frac{Y}{C_{1,1}}$ is in the ring $R[\frac{\epsilon}{a_{2,2}}, \frac{\epsilon}{b_{1,2}}, \frac{\epsilon}{b_{2,1}}, \frac{\epsilon}{C_{1,1}}]$.

****

To summarize and clarify the situation, this calculation shows that

$$FY = C_{1,1}x + C_{1,1}FZ_1 + FD_{1,0}D_{0,1} \cdot \epsilon,$$

where

(1) x is the polynomial in $Z[x_{i,j}]$ that results if ϵ is set to 0, and $\frac{x}{F} = e_{0,0}$.

(2) Z_1 is demonstrably in $R[\frac{\epsilon}{a_{2,2}}, \frac{\epsilon}{b_{1,2}}, \frac{\epsilon}{b_{2,1}}]$, and

(3) $\frac{\epsilon}{C_{1,1}}$ is the fraction that is adjoined to A at this stage. Therefore

$$E_{0,0} = \frac{Y}{C_{1,1}} = e_{0,0} + Z_1 + D_{1,0}D_{0,1}\frac{\epsilon}{C_{1,1}} \text{ is in } A.$$

****

Start with $y = d_{0,0}d_{1,1} - d_{0,1}d_{1,0}$. Multiply by $b_{1,1}$, to get

$$\begin{aligned} b_{1,1}y &= d_{1,1}\big(c_{0,0}c_{1,1} - c_{0,1}c_{1,0}\big) - b_{1,1}d_{0,1}d_{1,0} \\ &= c_{1,1}x_1 - u \end{aligned}$$

where $x_1 = d_{1,1}c_{0,0}$, and $u = c_{0,1}c_{1,0}d_{1,1} + b_{1,1}d_{0,1}d_{1,0}$.

$$\begin{aligned} b_{2,1}u &= b_{2,1}c_{0,1}c_{1,0}d_{1,1} + b_{2,1}b_{1,1}d_{0,1}d_{1,0} \\ &= b_{2,1}c_{0,1}c_{1,0}d_{1,1} + b_{1,1}d_{0,1}\big(c_{1,0}c_{2,1} - c_{1,1}c_{2,0}\big) \\ &= c_{1,0}v + c_{1,1}x_2 \end{aligned}$$

where $x_2 = b_{1,1}d_{0,1}c_{2,0}$ and $v = b_{2,1}d_{1,1}c_{0,1} + b_{1,1}d_{0,1}c_{2,1}$

$$\begin{aligned} b_{2,2}v &= b_{2,2}b_{2,1}d_{1,1}c_{0,1} + b_{2,2}b_{1,1}d_{0,1}c_{2,1} \\ &= b_{2,1}c_{0,1}\big(c_{1,1}c_{2,2} - c_{1,2}c_{2,1}\big) + b_{2,2}b_{1,1}d_{0,1}c_{2,1} \\ &= c_{1,1}x_3 + c_{2,1}w \end{aligned}$$

where $x_3 = b_{2,1}c_{0,1}c_{2,2}$ and $w = -b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}d_{0,1}$

$$\begin{aligned} b_{1,2}w &= -b_{1,2}b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}b_{1,2}d_{0,1} \\ &= -b_{1,2}b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}\big(c_{0,1}c_{1,2} - c_{0,2}c_{1,1})\big) \\ &= -b_{1,1}b_{2,2}c_{0,2}c_{1,1} + c_{0,1}c_{1,2}(-b_{1,2}b_{2,1} + b_{1,1}b_{2,2}) \\ &= -b_{1,1}b_{2,2}c_{0,2}c_{1,1} + c_{0,1}c_{1,2}a_{2,2}c_{1,1} \\ &= x_4c_{1,1} \end{aligned}$$

where $x_4 = -b_{1,1}b_{2,2}c_{0,2} + c_{0,1}c_{1,2}a_{2,2}$ divisible by $b_{1,2}$ because

$$\begin{aligned} x_4 &= a_{2,2}c_{0,1}c_{1,2} - b_{1,1}b_{2,2}c_{0,2} \\ &= a_{2,2}c_{0,1}c_{1,2} - a_{2,2}c_{1,1}c_{0,2} - b_{1,2}b_{2,1}c_{0,2} \\ &= a_{2,2}b_{1,2}d_{0,1} - b_{1,2}b_{2,1}c_{0,2} \end{aligned}$$

$$
\begin{aligned}
Fy &= b_{1,2}b_{2,2}b_{2,1}c_{1,1}x_1 \\
&- b_{1,2}b_{2,2}c_{1,1}x_2 \\
&- b_{1,2}c_{1,0}c_{1,1}x_3 \\
&- c_{1,0}c_{2,1}c_{1,1}x_4 \\
\\
&= c_{1,1}\big(b_{1,2}b_{2,2}b_{2,1}d_{1,1}c_{0,0} \\
&- b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} \\
&- b_{1,2}c_{1,0}(b_{2,1}c_{0,1}c_{2,2} - c_{1,0}c_{2,1}x_4)\big) \\
\\
&= c_{1,1}\big(b_{1,2}b_{2,2}b_{2,1}c_{0,0}d_{1,1} \\
&- b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} \\
&- b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} + c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}\big) \\
\\
&= c_{1,1}\big(b_{1,2}b_{2,2}b_{2,1}c_{0,0}d_{1,1} - b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} \\
&- b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} + c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}
\end{aligned}
$$

The next step. Start with $y = E_{0,0}E_{1,1} - E_{0,1}E_{1,0}$. Multiply by $C_{1,1}$, to get

$$\begin{aligned} C_{1,1}y &= E_{1,1}\big(D_{0,0}D_{1,1} - D_{0,1}D_{1,0}\big) - C_{1,1}E_{0,1}E_{1,0} \\ &= D_{1,1}x_1 - u \end{aligned}$$

where $x_1 = E_{1,1}D_{0,0}$, and $u = D_{0,1}D_{1,0}E_{1,1} + C_{1,1}E_{0,1}E_{1,0}$.

Multiply by $c_{2,1}$, to get

$$\begin{aligned} c_{2,1}u &= c_{2,1}D_{0,1}D_{1,0}E_{1,1} + c_{2,1}C_{1,1}E_{0,1}E_{1,0} \\ &= c_{2,1}D_{0,1}D_{1,0}E_{1,1} + C_{1,1}E_{0,1}\big(D_{1,0}D_{2,1} - D_{1,1}D_{2,0}\big) \\ &= D_{1,0}v + D_{1,1}x_2 \end{aligned}$$

where $x_2 = C_{1,1}E_{0,1}D_{2,0}$ and $v = c_{2,1}E_{1,1}D_{0,1} + C_{1,1}E_{0,1}D_{2,1}$

Multiply by $c_{2,2}$, to get

$$\begin{aligned} c_{2,2}v &= c_{2,2}c_{2,1}E_{1,1}D_{0,1} + c_{2,2}C_{1,1}E_{0,1}D_{2,1} \\ &= c_{2,1}D_{0,1}\big(D_{1,1}D_{2,2} - D_{1,2}D_{2,1}\big) + c_{2,2}C_{1,1}E_{0,1}D_{2,1} \\ &= D_{1,1}x_3 + D_{2,1}w \end{aligned}$$

where $x_3 = c_{2,1}D_{0,1}D_{2,2}$ and $w = -c_{2,1}D_{0,1}D_{1,2} + c_{2,2}C_{1,1}E_{0,1}$

Multiply by $c_{1,2}$, to get

$$\begin{aligned} c_{1,2}w &= -c_{1,2}c_{2,1}D_{0,1}D_{1,2} + c_{2,2}C_{1,1}c_{1,2}E_{0,1} \\ &= -c_{1,2}c_{2,1}D_{0,1}D_{1,2} + c_{2,2}C_{1,1}\big(D_{0,1}D_{1,2} - D_{0,2}D_{1,1})\big) \\ &= -C_{1,1}c_{2,2}D_{0,2}D_{1,1} + D_{0,1}D_{1,2}(-c_{1,2}c_{2,1} + C_{1,1}c_{2,2}) \\ &= -C_{1,1}c_{2,2}D_{0,2}D_{1,1} + D_{0,1}D_{1,2}b_{2,2}D_{1,1} \\ &= x_4D_{1,1} \end{aligned}$$

where $x_4 = -C_{1,1}c_{2,2}D_{0,2} + D_{0,1}D_{1,2}b_{2,2}$ divisible by $c_{1,2}$ because

$$\begin{aligned} x_4 &= b_{2,2}D_{0,1}D_{1,2} - C_{1,1}c_{2,2}D_{0,2} \\ &= b_{2,2}D_{0,1}D_{1,2} - b_{2,2}D_{1,1}D_{0,2} - c_{1,2}c_{2,1}D_{0,2} \\ &= b_{2,2}c_{1,2}E_{0,1} - c_{1,2}c_{2,1}D_{0,2} \end{aligned}$$

$$
\begin{aligned}
Cy &= c_{1,2}c_{2,2}c_{2,1}D_{1,1}x_1 \\
&- c_{1,2}c_{2,2}D_{1,1}x_2 \\
&- c_{1,2}D_{1,0}D_{1,1}x_3 \\
&- D_{1,0}D_{2,1}D_{1,1}x_4 \\
\\
&= D_{1,1}\big(c_{1,2}c_{2,2}c_{2,1}E_{1,1}D_{0,0} \\
&- c_{1,2}c_{2,2}C_{1,1}D_{2,0}E_{0,1} \\
&- c_{1,2}D_{1,0}c_{2,1}D_{0,1}D_{1,2} \\
&- D_{1,0}D_{2,1}x_4\big) \\
\\
&= D_{1,1}\big(c_{1,2}c_{2,2}c_{2,1}D_{0,0}E_{1,1} - c_{1,2}c_{2,2}C_{1,1}D_{2,0}E_{0,1} \\
&- c_{1,2}D_{1,0}c_{2,1}D_{0,1}D_{1,2} + D_{1,0}D_{2,1}C_{1,1}c_{2,2}D_{0,2} \\
&- D_{1,0}D_{2,1}D_{0,1}D_{1,2}b_{2,2}\big) \\
&= D_{1,1}\big(c_{1,2}c_{2,1}D_{0,0}D_{1,1}D_{2,2} - c_{1,2}c_{2,1}D_{0,0}D_{1,2}D_{2,1} \\
&- c_{2,2}C_{1,1}D_{2,0}D_{0,1}D_{1,2} + c_{2,2}C_{1,1}D_{2,0}D_{0,2}D_{1,1} \\
&- c_{1,2}D_{1,0}c_{2,1}D_{0,1}D_{1,2} + D_{1,0}D_{2,1}C_{1,1}c_{2,2}D_{0,2} - D_{1,0}D_{2,1}D_{0,1}D_{1,2}b_{2,2}\big)
\end{aligned}
$$