ELECTRICAL NETWORKS AND LIE THEORY

THOMAS LAM AND PAVLO PYLYAVSKYY

ABSTRACT. We introduce a new class of "electrical" Lie groups. These Lie groups, or more precisely their nonnegative parts, act on the space of planar electrical networks via combinatorial operations previously studied by Curtis-Ingerman-Morrow. The corresponding electrical Lie algebras are obtained by deforming the Serre relations of a semisimple Lie algebra in a way suggested by the star-triangle transformation of electrical networks. Rather surprisingly, we show that the type A electrical Lie group is isomorphic to the symplectic group. The nonnegative part $(EL_{2n})_{\geq 0}$ of the electrical Lie group is a rather precise analogue of the totally nonnegative subsemigroup $(U_n)_{\geq 0}$ of the unipotent subgroup of SL_n. We establish decomposition and parametrization results for $(EL_{2n})_{\geq 0}$, paralleling Lusztig's work in total nonnegativity, and work of Curtis-Ingerman-Morrow and de Verdière-Gitler-Vertigan for networks. Finally, we suggest a generalization of electrical Lie algebras to all Dynkin types.

1. INTRODUCTION

In this paper we consider the simplest of electrical networks, namely those that consist of only resistors. The electrical properties of such a network N are completely described by the *response matrix* $L(N)$, which computes the current that flows through the network when certain voltages are fixed at the boundary vertices of N. The study of the response matrices of *planar* electrical networks has led to a robust theory, see Curtis-Ingerman-Morrow [CIM], or de Verdière-Gitler-Vertigan [dVGV]. In 1899, Kennelly [Ken] described a local transformation (see Figure 3) of a network, called the *star-triangle* or $Y - \Delta$ transformation, which preserves the response matrix of a network. This transformation is one of the many places where a Yang-Baxter style transformation occurs in mathematics or physics.

Curtis-Ingerman-Morrow [CIM] studied the operations of *adjoining a boundary spike* and *adjoining a boundary edge* to (planar) electrical networks. Our point of departure is to consider these operations as one-parameter subgroups of a Lie group action. The star-triangle transformation then leads to an "electrical Serre relation" in the corresponding Lie algebra, which turns out to be a deformation of the Chevalley-Serre relation for $\mathfrak{sl}_n$:

$$\text{Serre relation: } [e, [e, e']] = 0 \qquad \text{electrical Serre relation: } [e, [e, e']] = -2e$$

The corresponding one-parameter subgroups satisfy a Yang-Baxter style relation which is a deformation of Lusztig's relation in total positivity:

Lusztig's relation: $u_i(a)u_j(b)u_i(c) = u_j(bc/(a+c))u_i(a+c)u_j(ab/(a+c))$

electrical relation: $u_i(a)u_j(b)u_i(c) = u_j(bc/(a+c+abc))u_i(a+c+abc)u_j(ab/(a+c+abc))$.

We study in detail the Lie algebra $\mathfrak{el}_{2n}$ with $2n$ generators satisfying the electrical Serre relation. An (electrically) nonnegative part $(EL_{2n})_{\geq 0}$ of the corresponding Lie group EL_{2n} acts on planar electrical networks with $n + 1$ boundary vertices, and one obtains a dense subset of all response matrices of planar networks in this way. This nonnegative part is a rather precise analogue of the totally nonnegative subsemigroup $(U_{2n+1})_{\geq 0}$ of the unipotent subgroup of SL_{2n+1}, studied in Lie-theoretic terms by Lusztig [Lus]. We show (Proposition 4.2) that $(EL_{2n})_{\geq 0}$ has a cell decomposition labeled by permutations $w \in S_{2n+1}$, precisely paralleling one of Lusztig's results for $(U_{2n})_{\geq 0}$ and reminiscent of the Bruhat decomposition. This can be considered an algebraic analogue of parametrization results in the theory of electrical networks [CIM, dVGV]. Surprisingly, EL_{2n} itself is isomorphic to the symplectic Lie group $Sp_{2n}(\mathbb{R})$ (Theorem 3.1). This semisimplicity does not in general hold for electrical Lie groups: EL_{2n+1} is not semisimple. We also describe (Theorem 4.10) the stabilizer lie algebra of the infinitesimal action of $\mathfrak{el}_{2n}$ on electrical networks.

While we focus on the planar case in this paper, we shall connect the results of this paper to the inverse problem for electrical networks on cylinders in a future paper. There we borrow ideas from representation theory such as that of crystals and R-matrices.

One obtains the types B and G electrical Serre relations by the standard technique of "folding" the type A electrical Serre relation. These lead to a new species of electrical Lie algebras $\mathfrak{e}_D$ defined for any Dynkin diagram D. Besides the above results for type A, there appear to be other interesting relations between these $\mathfrak{e}_D$ and simple Lie algebras. For example, $\mathfrak{eb}_2 := \mathfrak{e}_{B_2}$ is isomorphic to $\mathfrak{gl}_2$. We conjecture (Conjecture 5.2) that for a finite type diagram D, the dimension $\dim(\mathfrak{e}_D)$ is always equal to the dimension of the maximal nilpotent subalgebra of the semisimple Lie algebra $\mathfrak{g}_D$ with Dynkin diagram D, and furthermore (Conjecture 5.3) that $\mathfrak{e}_D$ is finite dimensional if and only if $\mathfrak{g}_D$ is finite-dimensional. We give the electrical analogue of Lusztig's relation for type B (see Berenstein-Zelevinsky [BZ]) where we observe similar positivity to the totally nonnegative case.

The most interesting case beyond finite Dynkin types is affine type A. The corresponding electrical Lie (semi)-group action on planar electrical networks is perhaps even more natural than that of EL_{2n}, since one can obtain *all* (rather than just a dense subset of) response matrices of planar networks in this way ([dVGV, CIM]), and furthermore the circular symmetry of the planar networks is preserved. We however do not attempt to address this case in the current paper.

2. Electrical networks

For more background on electrical networks, we refer the reader to [CIM, dVGV].

2.1. **Response matrix.** For our purposes, an electrical network is a finite weighted undirected graph N, where the vertex set is divided into the *boundary* vertices and the *interior* vertices. The weight $w(e)$ of an edge is to be thought of as the conductance of the corresponding resistor, and is generally taken to be a positive real number. A 0-weighted edge would be the same as having no edge, and an edge with infinite weight would be the same as identifying the endpoint vertices.

We define the *Kirchhoff matrix* $K(N)$ to be the square matrix with rows and columns labeled by the vertices of N as follows:

$$K_{ij} = \begin{cases} -\sum_{e \text{ joins } i \text{ and } j} w(e) & \text{for } i \neq j \\ \sum_{e \text{ incident to } i} w(e) & \text{for } i = j. \end{cases}$$

We define the *response matrix* $L(N)$ to be the square matrix with rows and columns labeled by the boundary vertices of N, given by the Schur-complement

$$L(N) = K/K_I$$

where K_I denotes the submatrix of K indexed by the interior vertices. The response matrix encodes all the electrical properties of N that can be measured from the boundary vertices.

2.2. **Planar electrical networks.** We shall usually consider electrical networks N embedded into a disk, so that the boundary vertices, numbered $1, 2, \ldots, n+1$, lie on the boundary of the disk. Given an odd integer $2k - 1$, for $k = 1, 2, \ldots, n + 1$, and a nonnegative real
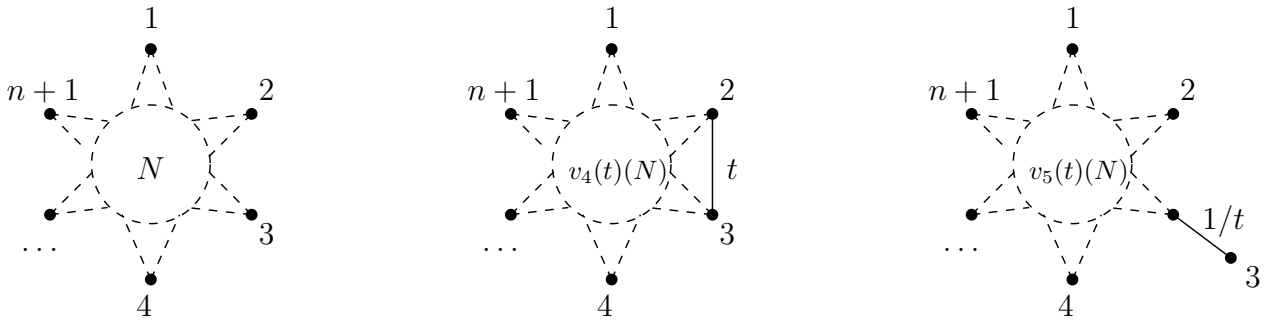

FIGURE 1. The operations $v_i(t)$ acting on a network N.

number t, we define $v_{2k-1}(t)(N)$ to be the electrical network obtained from N by adding a new edge from a new vertex v to k, with weight $1/t$, followed by treating k as an interior vertex, and the new vertex v as a boundary vertex (now named k).

Given an even integer $2k$, for $k = 1, 2, \ldots, n + 1$, and a nonnegative real number t, we define $v_{2k}(t)(N)$ to be the electrical network obtained from N by adding a new edge from k to $k + 1$ (indices taken modulo $n + 1$), with weight t.

The two operations are shown in Figure 1. In [CIM], these operations are called *adjoining a boundary spike*, and *adjoining a boundary edge* respectively. Our notation suggests, and we shall establish, that there is some symmetry between these two types of operations.

In [CIM, Section 8], it is shown that $L(v_i(a) \cdot N)$ depends only on $L(N)$, giving an operation $v_i(t)$ on response matrices. Denote by x_{ij} the entries of the response matrix, where $1 \leq i, j \leq n + 1$. In particular, we have $x_{ij} = x_{ji}$. Then if δ_{ij} denotes the Kronecker delta, we have

$$(1) \quad v_{2k-1}(t) : x_{ij} \mapsto x_{ij} - \frac{tx_{ik}x_{kj}}{tx_{kk} + 1} \quad \text{and} \quad v_{2k}(t) : x_{ij} \mapsto x_{ij} + (\delta_{ik} - \delta_{(i+1)k})(\delta_{jk} - \delta_{(j+1)k})t.$$

We caution that the parameter ξ in [CIM] is the inverse of our t in the odd case.

FIGURE 2. Series-parallel transformations.

2.3. **Electrically-equivalent transformations of networks.** Series-parallel transformations are shown in Figure 2. The following proposition is well-known and can be found for example in [dVGV].

Proposition 2.1. *Series-parallel transformations, removing loops, and removing interior degree 1 vertices, do not change the response matrix of a network.*
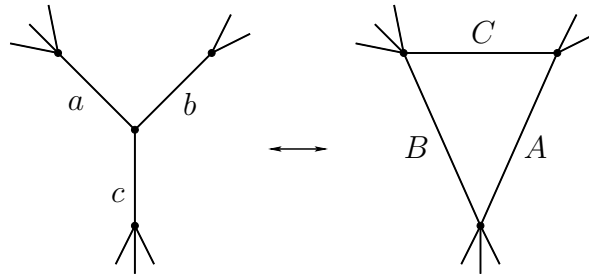

FIGURE 3. The $Y - \Delta$, or star-triangle, transformation.

The following theorem is attributed to Kennelly [Ken].

Proposition 2.2 ($Y - \Delta$ transformation). *Assume that parameters a,b,c and A,B,C are related by*
$$A = \frac{bc}{a+b+c}, \quad B = \frac{ac}{a+b+c}, \quad C = \frac{ab}{a+b+c},$$
or equivalently by
$$a = \frac{AB+AC+BC}{A}, \quad b = \frac{AB+AC+BC}{B}, \quad c = \frac{AB+AC+BC}{C}.$$
Then switching a local part of an electrical network between the two options shown in Figure 3 does not change the response matrix of the whole network.

3. THE ELECTRICAL LIE ALGEBRA $\mathfrak{el}_{2n}$

Let $\mathfrak{el}_{2n}$ be the Lie algebra $\mathfrak{el}_{2n}$ generated over $\mathbb{R}$ by e_i, $i = 1, \ldots, 2n$ subject to relations
- $[e_i, e_j] = 0$ for $|i - j| > 1$;
- $[e_i, [e_i, e_j]] = -2e_i$, $|i - j| = 1$.

Theorem 3.1. *The Lie algebra $\mathfrak{el}_{2n}$ is isomorphic to the real symplectic Lie algebra $\mathfrak{sp}_{2n}$.*

Proof. We identify the symplectic algebra $\mathfrak{sp}_{2n}$ with the space of $2n \times 2n$ matrices which in block form
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

satisfy $A = -D^T$, $B = B^T$ and $C = C^T$ (see [FH, Lecture 16]). Note that the subLie algebra consisting of the matrices where $B = C = 0$ is naturally isomorphic to $\mathfrak{gl}_n$.

Let $\varepsilon_i \in \mathbb{R}^n$ denote the standard basis column vector with a 1 in the i-th position. Let $a_1 = \varepsilon_1, a_2 = \varepsilon_1 + \varepsilon_2, a_3 = \varepsilon_2 + \varepsilon_3, \ldots, a_n = \varepsilon_{n-1} + \varepsilon_n$, and let $b_1 = \varepsilon_1, b_2 = \varepsilon_2, \ldots, b_n = \varepsilon_n$. For $1 \le i \le n$, define elements of $\mathfrak{sp}_{2n}$ by the formulae

$$e_{2i-1} = \begin{pmatrix} 0 & a_i \cdot a_i^T \\ 0 & 0 \end{pmatrix} \qquad e_{2i} = \begin{pmatrix} 0 & 0 \\ b_i \cdot b_i^T & 0 \end{pmatrix}.$$

We claim that this gives a symplectic representation ϕ of $\mathfrak{el}_{2n}$ which is an isomorphism of $\mathfrak{el}_{2n}$ with $\mathfrak{sp}_{2n}$. We first check that the relations of $\mathfrak{el}_{2n}$ are satisfied. It is clear from the block matrix form that $[e_{2i-1}, e_{2j-1}] = 0 = [e_{2i}, e_{2j}]$ for any i, j. Now,

$$[e_{2i-1}, e_{2j}] = \begin{pmatrix} (a_i \cdot a_i^T)(b_j \cdot b_j^T) & 0 \\ 0 & -(b_j \cdot b_j^T)(a_i \cdot a_i^T) \end{pmatrix}.$$

But by construction, $b_j^T \cdot a_i = 0 = a_i^T \cdot b_j$ unless $j = i$, or $j = i - 1$. Thus $[e_k, e_l] = 0$ unless $|k - l| = 1$. Finally,

$$[e_{2i-1}, [e_{2i-1}, e_{2i}]] = \begin{pmatrix} 0 & -2(a_i \cdot a_i^T)(b_i \cdot b_i^T)(a_i \cdot a_i^T) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2a_i \cdot a_i^T \\ 0 & 0 \end{pmatrix} = -2e_{2i-1}$$

using the fact that $a_i^T \cdot b_i = 1 = b_i^T \cdot a_i$. Similarly, one obtains $[e_k, [e_k, e_{k\pm1}]] = -2e_k$. Thus we have a symplectic representation $\phi : \mathfrak{el}_{2n} \to \mathfrak{sp}_{2n}$.

Now we show that this map is surjective. First we verify that the $\mathfrak{gl}_n \subset \phi(\mathfrak{el}_{2n})$. The non-zero commutators $[e_i, e_j]$ gives matrices of the form $\begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix}$ where A is a scalar multiple of one of the matrices $E_{1,1}$, or $E_{i,i} + E_{i+1,i}$ or $E_{i+1,i} + E_{i+1,i+1}$. Here $E_{i,j}$ denotes the $n \times n$ matrix with a single non-zero entry equal to one in the (i, j)-th position. All the matrices of the above form occur. It is easy to see that $\phi(\mathfrak{el}_{2n})$ must then contain the matrices where $A = E_{i,i}$, $E_{i,i+1}$, or $E_{i+1,i}$ for each i. But these matrices generate $\mathfrak{gl}_n$ as a Lie algebra. However, it is known [Hum, Proposition 8.4(d)] that for a semisimple Lie algebra L, one has $[L_\alpha, L_\beta] = L_{\alpha+\beta}$ when $\alpha, \beta, \alpha + \beta$ are all roots, and L_α denotes a root subspace. It follows easily from the explicit description of the root system of $\mathfrak{sp}_{2n}$ and the definition of e_i that every root subspace of $\mathfrak{sp}_{2n}$ is contained in $\phi(\mathfrak{el}_{2n})$, completing the proof.

To see that the map $\phi : \mathfrak{el}_{2n} \to \mathfrak{sp}_{2n}$ is injective, we note:

Lemma 3.2. *The dimension of $\mathfrak{el}_{2n}$ is $n(2n + 1)$.*

Proof. According to Lemma 5.1 the dimension of $\mathfrak{el}_{2n}$ is at most $n(2n + 1)$. On the other hand, we just saw that the map $\mathfrak{el}_{2n} \to \mathfrak{sp}_{2n}$ is surjective. The statement follows. $\square$

4. THE ELECTRICAL LIE GROUP EL_{2n}

Let EL_{2n} be the simply-connected (real) Lie group with Lie algebra $\mathfrak{el}_{2n}$. Let $u_i(a) = \exp(ae_i)$ denote the the one-parameter subgroups corresponding to the generators of $\mathfrak{el}_{2n}$.

Theorem 4.1. *The elements $u_i(a)$ satisfy the relations*
  (1) $u_i(a)u_i(b) = u_i(a + b)$,
  (2) $u_i(a)u_j(b) = u_j(b)u_i(a)$, *if* $|i - j| > 1$;
  (3) $u_i(a)u_j(b)u_i(c) = u_j(bc/(a + c + abc))u_i(a+c+abc)u_j(ab/(a + c + abc))$, *if* $|i-j| = 1$.

Proof. The first two relations are clear. For the third, observe that for each $1 \leq i \leq 2i - 1$, the elements e_i, e_{i+1}, and $[e_i, e_{i+1}]$ are the usual Chevalley generators of a Lie subalgebra isomorphic to $\mathfrak{sl}_2$. Thus we can verify the relation inside $SL_2(\mathbb{R})$:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + ab & a + c + abc \\ b & 1 + bc \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ bc/(a + c + abc) & 1 \end{pmatrix} \begin{pmatrix} 1 & a + c + abc \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ ab/(a + c + abc) & 1 \end{pmatrix}.$$

$\square$

Remark 4.1. There is a one-parameter deformation which connects the relation Theorem 4.1(3) with Lusztig's relation [Lus] in total positivity:

(2) $$u_i(a)u_j(b)u_i(c) = u_j(bc/(a + c))u_i(a + c)u_j(ab/(a + c)).$$

Consider the associative algebra $U_\tau(\mathfrak{el}_{2n})$ where the generators $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n}$ satisfy the following deformed Serre relations:

- $\varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i$ if $|i - j| > 1$;
- $\varepsilon_i \varepsilon_j \varepsilon_i = \tau \varepsilon_i + (\varepsilon_i^2 \varepsilon_j + \varepsilon_j \varepsilon_i^2)/2$, if $|i - j| = 1$.

This is a one-parameter family of algebras which at $\tau = 0$ reduces to $U(\mathfrak{n}^+)$, where $\mathfrak{sl}_{2n+1} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is the Cartan decomposition, while at $\tau = 1$ it gives the universal enveloping algebra $U(\mathfrak{el}_{2n})$ of the electrical Lie algebra. For $U_\tau(\mathfrak{el}_{2n})$, the "braid move" for the elements $\exp(a\varepsilon_i)$ then takes the form

$$u_i(a)u_j(b)u_i(c) = u_j(bc/(a + c + \tau abc))u_i(a + c + \tau abc)u_j(ab/(a + c + \tau abc)) \quad \text{if } |i - j| = 1.$$

At $\tau = 0$ this reduces to (2), and at $\tau = 1$ it is the relation Theorem 4.1(3).

4.1. **Nonnegative part of EL_{2n}.** Define the *nonnegative* part $(EL_{2n})_{\geq 0}$ of EL_{2n}, or equivalently the *electrically nonnegative* part of Sp_{2n}, to be the subsemigroup generated by the $u_i(a)$ with nonnegative parameters a.

For a reduced word $\mathbf{i} = i_1 \ldots i_\ell$ of $w \in S_{2n+1}$ denote by $E(w) \subset (EL_{2n})_{\geq 0}$ the image of the map

$$\psi_{\mathbf{i}} : (a_1, \ldots, a_\ell) \in \mathbb{R}_{>0}^\ell \mapsto u_{i_1}(a_1) \ldots u_{i_\ell}(a_\ell).$$

It follows from the relations in Theorem 4.1 that the set $E(w)$ depends only on w, and not on the chosen reduced word. The following proposition is similar to [Lus, Proposition 2.7] which gives a decomposition $U_{\geq 0} = \sqcup_w U_{\geq 0}^w$ of the totally nonnegative part of the unipotent group.

Proposition 4.2. *The sets $E(w)$ are disjoint and cover the whole $(EL_{2n})_{\geq 0}$. Each of the maps $\psi_{\mathbf{i}} : \mathbb{R}_{>0}^\ell \to E(w)$ is a homeomorphism.*

Proof. Using Theorem 4.1, we can rewrite any product of generators $u_i(a)$ by performing braid moves similar to those in the symmetric group S_{2n+1}. Any product of the generators can be transformed into a product that corresponds to a reduced word in S_{2n+1}, and thus it belongs to one of the $E(w)$.

If the map $\psi_{\mathbf{i}}$ is not injective for some reduced word $\mathbf{i}$, then we can find two reduced products

$$u_{i_1}(a_1) \ldots u_{i_\ell}(a_\ell) = u_{i_1}(b_1) \ldots u_{i_\ell}(b_\ell)$$

6

for two ℓ-tuples of positive numbers such that $a_1 \neq b_1$. Without loss of generality we can assume $a_1 > b_1$, and thus

$$u_{i_1}(a_1 - b_1)u_{i_2}(a_2)\ldots u_{i_\ell}(a_\ell) = u_{i_2}(b_2)\ldots u_{i_\ell}(b_\ell).$$

This shows that two different $E(w)$-s have non-empty intersection. Thus it suffices to show that the latter is impossible.

Furthermore, it suffices to prove that the top cell corresponding to the longest element w_0 does not intersect any other cell. Indeed, if any two cells intersect, by adding some extra factors to both we can lift it to the top cell intersecting one of the other cells. Assume we have a product of the form

(3) $$u = [u_1(a_1)][u_2(a_2)u_1(a_3)]\ldots[u_k(a_{\ell-k+1})\ldots u_1(a_\ell)]$$

where $\ell = \binom{k}{2}$. Let $\Phi(u) \in Sp_{2n}$ be the image of u in Sp_{2n} under the natural map $EL_{2n} \to Sp_{2n}$ induced by the map $\phi : \mathfrak{el}_{2n} \to \mathfrak{sp}_{2n}$ of Theorem 3.1. We argue that the positive parameters $a_{\ell-k+1}, \ldots, a_\ell$ can be recovered uniquely from just looking at the $n + k/2$-th row of $\Phi(u)$ for k even, or the $(k+1)/2$-th row for k odd. Furthermore, if exactly one of them is equal to zero, then the same calculation will tell us that.

Indeed, assume k is odd. Then the $n + (k+1)/2$-th entry in the $(k+1)/2$-th row is just equal to $a_{\ell-k+1}$. Next, once we know $a_{\ell-k+1}$, we can use $(k-1)/2$-th entry of the same row to recover $a_{\ell-k+2}$, after which we can use $n + (k-1)/2$ entry to recover $a_{\ell-k+3}$, and so on. For each step we solve a linear equation where the parameter we divide by is strictly positive as long as all previous parameters a_i recovered are positive. For example, let $n = 2$ and take the product

$$u_1(a_1)u_2(a_2)u_1(a_3)u_3(a_4)u_2(a_5)u_1(a_6).$$

Then the second row of this product is $(a_4 a_5, 1, a_4 + a_4 a_5 a_6, a_4)$. The last entry tells us a_4, then from the first entry we solve for a_5, then from the third entry we solve for a_6. The case of even k is similar, the order in which we have to read the entries of the $n + k/2$-th row in this case is $k/2$-th, $n + k/2$-th, $k/2 - 1$-th, $n + k/2 - 1$-th, etc. For example, in the product

$$u_1(a_1)u_2(a_2)u_1(a_3)u_3(a_4)u_2(a_5)u_1(a_6)u_4(a_7)u_3(a_8)u_2(a_9)u_1(a_{10}),$$

the last row is $(a_7 a_8 a_9, a_7, a_7 a_8 + a_7 a_8 a_9 a_{10}, 1 + a_7 a_8)$. By looking at second, last, first and third entries we can solve for a_7, a_8, a_9 and a_{10} one after the other.

Now we are ready to complete the argument. Assume that $E(w_0)$ intersects some other $E(v)$. For any reduced word of w_0, one can find a subword which is a reduced word for v, so that we have

$$u = [u_1(a_1)][u_2(a_2)u_1(a_3)]\ldots[u_k(a_{\ell-k+1})\ldots u_1(a_\ell)]$$
$$= [u_1(b_1)][u_2(b_2)u_1(b_3)]\ldots[u_k(b_{\ell-k+1})\ldots u_1(b_\ell)],$$

where the a_i are all positive, but the b_i are nonnegative, with at least one zero. The above algorithm of recovering the a_i's will arrive at a contradiction. Thus the $E(w_0)$ is disjoint from all other $E(v)$.

It remains to show that $\phi_{\mathbf{i}}^{-1}$ is continuous for any reduced word. If $\mathbf{i}$ is the reduced word of w_0 used in (3), this follows from the algorithm above: the a_i depend continuously on the matrix entries of $\Phi(u)$. But any two reduced words are connected by braid and commutation moves, so it follows from the (continuously invertible) formulae in Theorem 4.1 that $\phi_{\mathbf{i}}^{-1}$ is continuous for any reduced word of w_0. But for any other $v \in S_{2n+1}$, a reduced word $\mathbf{j}$ for v

can be found as an initial subword of some reduced word $\mathbf{i}$ for w_0. The map $\phi_{\mathbf{j}}^{-1}$ can then be expressed as a composition of: right multiplication by a fixed element of EL_{2n}, the map $\phi_{\mathbf{i}}^{-1}$, and the projection from $\mathbb{R}_{>0}^{\ell(w_0)}$ to $\mathbb{R}_{>0}^{\ell(v)}$, all of which are continuous. $\qquad\square$

4.2. **Action on electrical networks.** Let $\mathcal{P}(n+1) \subset \mathbb{R}^{(n+1)^2}$ denote the set of response matrices of planar electrical networks with $n+1$ boundary vertices. In [CIM], it is shown that $\mathcal{P}(n+1)$ is exactly the set of symmetric, circular totally-nonnegative $(n+1) \times (n+1)$ matrices. In [dVGV, Théorème 4], it is shown that $\mathcal{P}(n+1)$ can be identified with the set of planar electrical networks with $n+1$ boundary vertices modulo the local transformations of Section 2.3. Let N_0 denote the empty network (with $n+1$ boundary vertices) and let $L_0 = L(N_0)$ denote the zero matrix.

Theorem 4.3. *The nonnegative part $(EL_{2n})_{\geq 0}$ of the electrical group acts on $\mathcal{P}(n+1)$ via:*

$$u_i(a) \cdot L(N) = L(v_i(a)(N)).$$

Proof. For a single generator $u_i(a)$, the stated action is well-defined because it can be described explicitly on the level of response matrices. The formulae for $L(v_i(a)(N))$ in terms of $L(N)$ is given by the equation (1).

We first show that the relations of Theorem 4.1 hold for this action. Relation (1) follows from the series-parallel relation for networks. Relation (2) is immediate: the corresponding networks are identical without any transformations. Relation (3) follows from the $Y - \Delta$ transformation (see Example 4.4).

But now suppose we have two different expressions for $u \in (EL_{2n})_{\geq 0}$ in terms of generators. Then using Relations (1)-(3) of Theorem 4.1, we may assume that both expressions are products corresponding to a reduced word. By Proposition 4.2, the two products come from reduced words for the same $w \in S_{2n+1}$. It follows that they are related by relations (1)-(3). $\qquad\square$

Example 4.4. The products $u_3(a)u_4(b)u_3(c)$ and $u_4(\frac{bc}{a+c+abc})u_3(a + c + abc)u_4(\frac{ab}{a+c+abc})$ act on a network in exactly the same way, as shown in Figure 4.

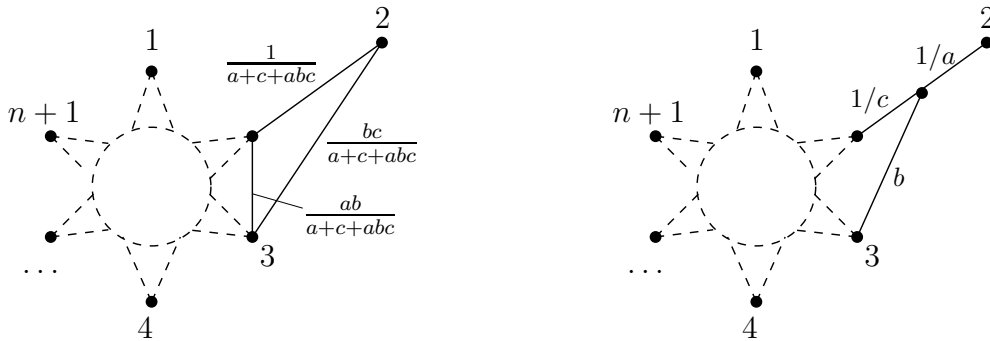

FIGURE 4.

A permutation $w = w(1)w(2) \cdots w(2n+1) \in S_{2n+1}$ is *efficient* if
 (1) $w(1) < w(3) < \cdots < w(2n+1)$ and $w(2) < w(4) < \cdots < w(2n)$
 (2) $w(1) < w(2)$, $w(3) < w(4)$, …, $w(2n-1) < w(2n)$.

Recall the *left weak order* of permutations is given by $w \preceq v$ if and only if there is a u so that $uw = v$ and $\ell(v) = \ell(u) + \ell(w)$. It is a standard fact that $w \preceq v$ if and only if whenever $w(i) > w(j)$ and $i < j$ then $v(i) > v(j)$. Thus the set of efficient permutations has a maximum in left weak order, namely $w = 1(n+2)2(n+3)\cdots(n)(2n+1)(n+1)$ with length $n(n-1)/2$.

Theorem 4.5. *Let $w \in S_{2n+1}$. The map $\Theta_w : E(w) \to \mathcal{P}(n+1)$ given by $\Theta_w(u) = u \cdot L_0$ is injective if and only if w is efficient. We have $\mathrm{image}(\Theta_w) \cap \mathrm{image}(\Theta_v) = \emptyset$ for $w \neq v$ both efficient. If w is not efficient, there is a unique efficient v such that $\mathrm{image}(\Theta_w) = \mathrm{image}(\Theta_v)$.*

Proof. Let $w^* \in S_{2n+1}$ be the efficient permutation of maximal length. Then a possible reduced word for w^* is $(n+1)\cdots(46\cdots(2n-2))(35\cdots(2n-1))(246\cdots(2n))$. The graph obtained by taking the corresponding $u_i(a)$ and acting on the empty network is exactly the "standard graph" of [CIM, Section 7] or the graph C_N of [dVGV]. In particular, Θ_{w^*} is injective by [CIM, Theorem 2] or [dVGV, Théorème 3]. Suppose w is an arbitrary efficient permutation. Then since $w^* = uw$ for some $u \in S_{2n+1}$, if Θ_w is not injective then Θ_{w^*} is not injective as well, so we conclude that Θ_w is injective.

For a pair (i, j) with $1 \leq i < j \leq n+1$, let us say that a network N is (i, j)-*connected* if we can find a disjoint set of paths $p_1, p_2, ..., p_{\lfloor (j-i+1)/2 \rfloor}$ so that for each k, the path p_k connects boundary vertex $i + k - 1$ to boundary vertex $j - k + 1$ without passing through any other boundary vertex. This is a special case of the connections of circular pairs (P, Q) of [CIM]. Let N_w be a graph constructed from some reduced word of an efficient w. We observe that N_w is (i, j)-connected if and only if $w(2i) > w(2j - 1)$. For example, the first
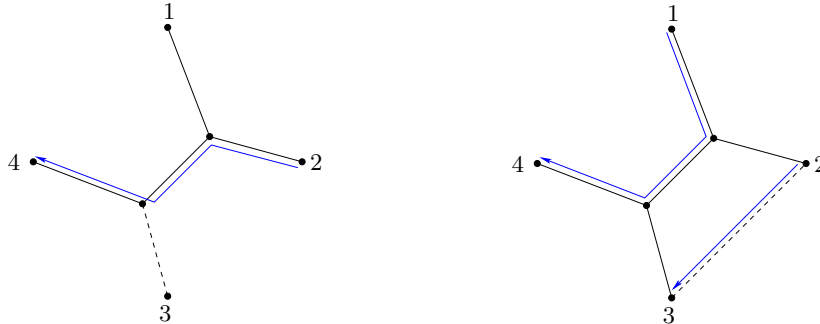

FIGURE 5.

network in Figure 5 corresponds to $w = s_5 s_3 s_6 s_4 s_2 = (1, 3, 5, 2, 7, 4, 6)$. We see that it is $(2, 4)$-connected, which agrees with $w(4) = 6 > 5 = w(7)$. If the dashed edge is not there, we have $w = s_3 s_6 s_4 s_2 = (1, 3, 5, 2, 4, 7, 6)$ and $w(4) = 5 < 6 = w(7)$ in agreement with the network not being $(2, 4)$-connected. Similarly, the second network in Figure 5 corresponds to $w = s_4 s_5 s_3 s_6 s_4 s_2 = (1, 3, 5, 7, 2, 4, 6)$ with the dashed edge and to $w = s_5 s_3 s_6 s_4 s_2 = (1, 3, 5, 2, 7, 4, 6)$ without. In the first case it is $(1, 4)$-connected, in the second it is not, in agreement with relative order of $w(2)$ and $w(7)$.

Note that the inversions $w(2i) > w(2j - 1)$ are exactly the possible inversions of an efficient permutation. Since w is determined by its inversions, it follows that the set of (i, j)-connections of N_w determines w, and that $\mathrm{image}(\Theta_w) \cap \mathrm{image}(\Theta_v) = \emptyset$ for $w \neq v$ both efficient.

Suppose w is not efficient. Then it is easy to see that w has a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ where either (1) i_ℓ is odd, or (2) i_ℓ is even and $i_\ell = i_{\ell-1} \pm 1$. But $u_i(a) \cdot L_0 = L_0$ for odd i since all the boundary vertices are still disconnected in $u_i(a) \cdot N_0$, and $u_{i\pm1}(a)u_i(b) \cdot L_0 = u_i(1/a + b)L_0$, using the series-transformation (Proposition 2.1). It follows that Θ_w is not injective. Furthermore, $\text{image}(\Theta_w) = \text{image}(\Theta_v)$, where v is obtained from w by recursively (1) removing i_ℓ from a reduced word of w if i_ℓ is odd, or (2) changing the last two letters $(i_\ell \pm 1)i_\ell$ to i_ℓ when i_ℓ is even. An efficient v obtained in this way must be unique, since $\text{image}(\Theta_v) \cap \text{image}(\Theta_{v'}) = \emptyset$ for efficient $v \neq v'$. $\qquad\square$

Corollary 4.6. *The number of efficient $w \in S_{2n+1}$ is equal to the Catalan number $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$.*

Proof. It is clear that $(2i, 2j+1)$ is an inversion of w only if $(2i, 2j-1)$ and $(2i+2, 2j+1)$ are also inversions, and this characterizes inversion sets of efficient w. Thus the set of efficient $w \in S_{2n+1}$ is in bijection with the lower order ideals of the positive root poset of $\mathfrak{sl}_{n+1}$ which is well known to be enumerated by the Catalan number [FR]. $\qquad\square$

Corollary 4.7. $(EL_{2n})_{\geq 0} \cdot L_0$ *is dense in* $\mathcal{P}(n+1)$.

Proof. Follows from Theorem 4.5 and [dVGV, Théorème 5]. $\qquad\square$

4.3. **Stabilizer.**

Lemma 4.8. *The stabilizer subsemigroup of* $(EL_{2n})_{\geq 0}$ *acting on the zero matrix L_0 is the subsemigroup generated by* $u_{2i+1}(a)$.

Proof. It is clear that $u_{2i+1}(a)$ lies in the stabilizer. But the action of any $u_{2i}(a)$ will change the connectivity of the network, and it is impossible to return to trivial connectivity by adding more edges, or by relations. $\qquad\square$

The semigroup stabilizer is too small in the sense that it does not detect the relations $u_{i\pm1}(a)u_i(b) \cdot L_0 = u_i(1/a + b)L_0$ used in the proof of Theorem 4.5. We shall calculate the stabilizer subalgebra of the corresponding infinitesimal action of the Lie algebra $\mathfrak{el}_{2n}$, which will in particular give an algebraic explanation of these relations. The reason we do not work with the whole Lie group EL_{2n} is threefold: (1) non-positive elements of EL_{2n} will produce networks that are "virtual", that is, have negative edge weights; (2) the topology of EL_{2n} means that to obtain an action one cannot just check the relations of Theorem 4.1; (3) when the parameters are non-positive, the relation Theorem 4.1(3) develops singularities.

To describe the infinitesimal action of $\mathfrak{el}_{2n}$, we give derivations of $\mathbb{R}[x_{ij}]$, the polynomial ring in $(n+1)(n+2)/2$ variables x_{ij} where $1 \leq i, j \leq n+1$ and we set $x_{ij} = x_{ji}$.

Proposition 4.9. *The electrical Lie algebra* $\mathfrak{el}_{2n}$ *acts on* $\mathbb{R}[x_{ij}]$ *via derivations as follows:*

$$(4) \qquad\qquad e_{2i} \mapsto \partial_{ii} + \partial_{i+1,i+1} - \partial_{i,i+1}$$

$$(5) \qquad\qquad e_{2i-1} \mapsto -\sum_{1 \leq p \leq q \leq n+1} x_{ip} x_{iq} \partial_{pq}$$

Proof. These formulae can be checked by directly verifying the defining relations of $\mathfrak{el}_{2n}$. Alternatively, they can be deduced by differentiating the formulae (1). $\qquad\square$

We calculate that

$$(6) \qquad [e_{2i}, e_{2i-1}] \mapsto -x_{ii}\partial_{ii} + x_{i,i+1}\partial_{i+1,i+1} + \sum_{1 \leq p \leq n+1} (x_{i+1,p}\partial_{i+1,p} - x_{ip}\partial_{ip}).$$

Theorem 4.10. *The stabilizer subalgebra $\mathfrak{el}_{2n}^0$, at the zero matrix L_0, of the infinitesimal action of $\mathfrak{el}_{2n}$ on the space of response matrices is generated by e_i for i odd, and $[e_{2i-1}, e_{2i}]$ for $i = 1, 2, \ldots, n$.*

Proof. The fact the stated elements lie in $\mathfrak{el}_{2n}^0$ follows from (5) and (6), since $x_{ij} = 0$ at the zero matrix. By Lemma 3.2, the elements e_α in the proof of Lemma 5.1 form a basis of $\mathfrak{el}_{2n}$. Write $\alpha_{ij} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $1 \leq i \leq j \leq 2n$ to denote the positive roots of $\mathfrak{sl}_{2n+1}$. Then we know that $e_{\alpha_{i,i}} \in \mathfrak{el}_{2n}^0$ for i odd, and $e_{\alpha_{i,i+1}} \in \mathfrak{el}_{2n}^0$ for each i. It follows easily that $e_{\alpha_{i,j}} \in \mathfrak{el}_{2n}^0$ for every pair $1 \leq i \leq j \leq 2n$ where at least one of i and j are odd. It follows that

$$\dim_\mathbb{R}(\mathfrak{el}_{2n}) - \dim_\mathbb{R}(\mathfrak{el}_{2n}^0) \leq \#\{(i,j) \mid 1 \leq i \leq j \leq 2n \text{ and } i \text{ and } j \text{ are even}\} = n(n+1)/2.$$

By Theorem 4.5 the action of $(EL_{2n})_{\geq 0}$ on the zero matrix L_0 gives a space of dimension $n(n+1)/2$. It follows that the above inequality is an equality, and that $\mathfrak{el}_{2n}^0$ is generated by the stated elements. $\qquad\square$

Note that a basis for $\mathfrak{el}_{2n}/\mathfrak{el}_{2n}^0$ is given by the $e_{\alpha_{i,j}}$'s where i and j are both even. These $\alpha_{i,j}$'s are exactly the inversions of efficient permutations (cf. proof of Theorem 4.5).

5. ELECTRICAL LIE ALGEBRAS OF FINITE TYPE

5.1. **Dimension.** Let D be a Dynkin diagram of finite type and let $A = (a_{ij})$ be the associated Cartan matrix. To each node i of D associate a generator e_i, and define $\mathfrak{e}_D$ to be the Lie algebra generated over $\mathbb{R}$ by the e_i modulo for each $i \neq j$ the relations $ad(e_i)^{1-a_{ij}}(e_j) = 0$ for $a_{ij} \neq -1$ and $ad(e_i)^2(e_j) = -2e_i$ for $a_{ij} = -1$.

These "electrical Serre relations" can be deduced from the type A electrical Serre relations of Section 3 by *folding*. Namely, the relation for an edge of multiplicity two $(a_{ij} = -2)$ can be obtained by finding the relation for the elements e_2 and $e_1 + e_3$ inside $\mathfrak{el}_3$. Similarly, the $\mathfrak{e}_{G_2}$ relation $(a_{ij} = -3)$ can be obtained by finding the relation for the elements $e_1 + e_2 + e_3$ and e_4 inside $\mathfrak{e}_{D_4}$, where e_4 corresponds to the node of valency three in D_4.

Lemma 5.1. *The dimension of the Lie algebra $\mathfrak{e}_D$ does not exceed the number of positive roots in the finite root system associated to D.*

Proof. Consider the nilpotent subalgebra $\mathfrak{u}_D$ of the usual Lie algebra associated to D. It is well known that it is generated by Chevalley generators $\tilde{e}_i$ subject to the Serre relations: $ad(\tilde{e}_i)^{1-a_{ij}}(\tilde{e}_j) = 0$ for each $i \neq j$.

The Lie algebra $\mathfrak{u}_D$ has a basis $\tilde{e}_\alpha$ labeled by positive roots $\alpha \in R^+$ of the root system, where the $\tilde{e}_i$ correspond to simple roots. Let us fix an expression

$$\tilde{e}_\alpha = [\tilde{e}_{i_1}, [\tilde{e}_{i_2}, [\ldots, [\tilde{e}_{i_{l-1}}, \tilde{e}_{i_l}]\ldots]]]$$

of shortest possible length for each $\tilde{e}_\alpha$, and define the corresponding

$$(7) \qquad\qquad e_\alpha = [e_{i_1}, [e_{i_2}, [\ldots, [e_{i_{l-1}}, e_{i_l}]\ldots]]]$$

11

in $\mathfrak{e}_D$. We claim that the e_α span $\mathfrak{e}_D$. Indeed, it is enough to show that any expression $[e_{j_1}, [e_{j_2}, [\ldots, [e_{j_{\ell-1}}, e_{j_\ell}]\ldots]]]$ lies in the linear span of the e_α-s. Assume otherwise, and take a counterexample of smallest possible total length ℓ. Let us define $\hat{e}_\alpha$ in the free Lie algebra $\mathfrak{f}_D$ with generators $\hat{e}_i$ using (7). Then in $\mathfrak{f}_D$ we have a relation of the form

$$[\hat{e}_{j_1}, [\hat{e}_{j_2}, [\ldots, [\hat{e}_{j_{\ell-1}}, \hat{e}_{j_\ell}]\ldots]]] - \sum_\alpha c_\alpha \hat{e}_\alpha = \hat{x} \in I$$

where I denotes the ideal generated by the Serre relations, so that $\mathfrak{u}_D = \mathfrak{f}_D/I$. Now $\mathfrak{f}_D$ is naturally $\mathbb{Z}$-graded, and I is a graded ideal, so we may assume all terms in the relation are homogeneous with the same degree. Now replacing each instance of the Serre relation in x with the corresponding electrical Serre relation gives us a relation

$$[e_{j_1}, [e_{j_2}, [\ldots, [e_{j_{\ell-1}}, e_{j_\ell}]\ldots]]] - \sum_\alpha c_\alpha e_\alpha = x$$

in $\mathfrak{e}_D$, where x is a sum of terms of the form $[e_{k_1}, [e_{k_2}, [\ldots, [e_{k_{\ell-1}}, e_{k_{\ell'}}]\ldots]]]$ where $\ell' < \ell$, and thus by assumption lies in the span of the e_α-s. The statement of the lemma follows. $\square$

Conjecture 5.2. *The dimension of $\mathfrak{e}_D$ coincides with the number of positive roots in the root system of D.*

One can also define the Lie algebras $\mathfrak{e}_D$ for any Dynkin diagram D, finite type or not.

Conjecture 5.3. *The Lie algebra $\mathfrak{e}_D$ is finite dimensional if and only if D is of finite type.*

5.2. **Examples.** We illustrate Conjecture 5.2 with some examples. For electrical type A_{2n} it has already been verified in Theorem 3.1.

5.2.1. *Electrical B_2.* Consider the case when D has two nodes connected by a double edge. In that case we denote by $\mathfrak{e}_D = \mathfrak{eb}_2$ the Lie algebra generated by two generators e and f subject to the relations

$$[e, [e, [e, f]]] = 0, \quad [f, [f, e]] = -2f.$$

Lemma 5.4. *The Lie algebras $\mathfrak{eb}_2$ and $\mathfrak{gl}_2$ are isomorphic.*

Proof. Consider the map

$$e \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

One easily checks that it is a Lie algebra homomorphism, and that it is surjective. By Lemma 5.1 we know the dimension of $\mathfrak{eb}_2$ is at most four, so the map must be an isomorphism. $\square$

5.2.2. *Electrical G_2.* Consider the Lie algebra $\mathfrak{eg}_2$ generated by two generators e and f subject to the relations

$$[e, [e, [e, [e, f]]]] = 0, \quad [f, [f, e]] = -2f.$$

Lemma 5.5. *The Lie algebra $\mathfrak{eg}_2$ is six-dimensional.*

Proof. According to Lemma 5.1 the elements $e, f, [ef], [e[ef]], [e[e[ef]]], [f[e[e[ef]]]]$ form a spanning set for $\mathfrak{eg}_2$. Thus it remains to check that they are linearly independent. This is easily done inside the following faithful representation of $\mathfrak{eg}_2$ in $\mathfrak{gl}_4$:

$$e \mapsto \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$\square$

5.2.3. *Electrical C_3.* Consider the Lie algebra $\mathfrak{ec}_3$ generated by three generators e, f and g subject to the relations

$$[e, [e, [e, f]]] = 0, \quad [f, [f, e]] = -2f, \quad [f, [f, g]] = -2f, \quad [g, [g, f]] = -2g, \quad [e, g] = 0.$$

Lemma 5.6. *The Lie algebra $\mathfrak{ec}_3$ is nine-dimensional.*

Proof. We consider two representations of $\mathfrak{ec}_3$: one inside $\mathfrak{sl}_9$ and one inside $\mathfrak{sl}_2$. Let E_{ij} denote a matrix with a 1 in the (i, j)-th position and 0's elsewhere. For the first representation, we define:

$$e \mapsto E_{42} + E_{54} + E_{76} + E_{87} + E_{89}$$
$$f \mapsto 2E_{24} - 2E_{26} + 2E_{41} + 2E_{45} - E_{47} - 2E_{63} + E_{67} + E_{98}$$
$$g \mapsto -E_{36} - E_{62} - E_{74} - E_{85} + E_{89}$$

For the second representation we define

$$e \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad g \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

One verifies directly that these are indeed representations of $\mathfrak{ec}_3$, and the direct sum of these two representations is faithful, from which the dimension is easily calculated. In fact, the first representation is simply the adjoint representation of $\mathfrak{ec}_3$, in the basis

$$e, f, g, [ef], [e[ef]], [fg], [e[fg]], [e[e[fg]]], [f[e[e[fg]]]],$$

and $\mathfrak{ec}_3$ has a one-dimensional center which acts non-trivially in the second representation. $\square$

5.3. **$Y - \Delta$ transformation of type B.** Let e and f be the generators of $\mathfrak{eb}_2$ as before, and denote by $u(t) = \exp(te)$ and $v(t) = \exp(tf)$ the corresponding one-parameter subgroups. The following proposition is a type B analog of the star-triangle transformation (Proposition 2.2).

Proposition 5.7. *We have*

$$u(t_1)v(t_2)u(t_3)v(t_4) = v(p_1)u(p_2)v(p_3)u(p_4),$$

where

$$p_1 = \frac{t_2 t_3^2 t_4}{\pi_2}, \quad p_2 = \frac{\pi_2}{\pi_1}, \quad p_3 = \frac{\pi_1^2}{\pi_2}, \quad p_4 = \frac{t_1 t_2 t_3}{\pi_1},$$

where

$$\pi_1 = t_1 t_2 + (t_1 + t_3)t_4 + t_1 t_2 t_3 t_4, \quad \pi_2 = t_1^2 t_2 + (t_1 + t_3)^2 t_4 + t_1 t_2 t_3 t_4 (t_1 + t_3).$$

13

Proof. Direct calculation inside GL_2, using Lemma 5.4. We have

$$u(t) = \begin{pmatrix} e^t & e^t t \\ 0 & e^t \end{pmatrix}, \quad v(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

and both sides of the equality are equal to

$$\begin{pmatrix} e^{t_1+t_3}(1 + t_3 t_4 + t_1(t_2 + t_4 + t_2 t_3 t_4)) & e^{t_1+t_3}(t_1 + t_3 + t_1 t_2 t_3) \\ e^{t_1+t_3}(t_2 + t_4 + t_2 t_3 t_4) & e^{t_1+t_3}(1 + t_2 t_3) \end{pmatrix}.$$

$\square$

Remark 5.1. One can consider a one-parameter family of deformations of the above formulas by taking

$$\pi_1 = t_1 t_2 + (t_1 + t_3)t_4 + \tau t_1 t_2 t_3 t_4, \quad \pi_2 = t_1^2 t_2 + (t_1 + t_3)^2 t_4 + \tau t_1 t_2 t_3 t_4 (t_1 + t_3).$$

At $\tau = 0$ this specializes to the transformation found by Berenstein and Zelevinsky in [BZ, Theorem 3.1]. Furthermore, just like the transformation in [BZ], this deformation is given by *positive* rational formulae.

References

[BZ] A. Berenstein and A. Zelevinsky: Total positivity in Schubert varieties, *Comment. Math. Helv.*, 72 (1997), no. 1, 128–166.

[CIM] E.B. Curtis, D. Ingerman, and J.A. Morrow: Circular planar graphs and resistor networks, *Linear Algebra Appl.*, 283 (1998), no. 1-3, 115–150.

[dVGV] Y.C. de Verdière, I. Gitler, and D. Vertigan: Réseaux électriques planaires. II, *Comment. Math. Helv.*, 71(1) (1996), 144–167

[FH] W. Fulton and J. Harris: Representation theory. A first course., *Graduate Texts in Mathematics*, 129, Springer-Verlag, New York, 1991. xvi+551 pp.

[Hum] J.E. Humphreys: Reflection groups and Coxeter groups, *Cambridge Studies in Advanced Mathematics*, 29, Cambridge University Press, Cambridge, 1990. xii+204 pp.

[FR] S. Fomin and N. Reading: Root systems and generalized associahedra, *Geometric combinatorics*, 63–131, IAS/Park City Math. Ser., 13.

[Ken] A.E. Kennelly: Equivalence of triangles and stars in conducting networks, *Electrical World and Engineer*, 34 (1899), 413–414.

[Lus] G. Lusztig: Total positivity in reductive groups, *Lie theory and geometry*, 531–568, Progr. Math., 123, Birkhäuser Boston, Boston, MA, 1994.

E-mail address: `tfylam@umich.edu`

Department of Mathematics, University of Michigan, 530 Church St., Ann Arbor, MI 48109 USA

E-mail address: `ppylyavs@umn.edu`

Department of Mathematics, University of Minnesota, 206 Church St. SE, Minneapolis, MN 55455 USA