Fouier analysis of waveforms of periodic sequences

We consider the waveforms generated by periodic integer sequences $a_n = mn$ where m is an integer greater than 1. (We use the term *periodic integer sequence* here to refer to an increasing positive integer sequence with the property that there is an integer P such that, for all k is in the sequence, $k + P$ is in the sequence; this property results in a periodic waveform using our method).

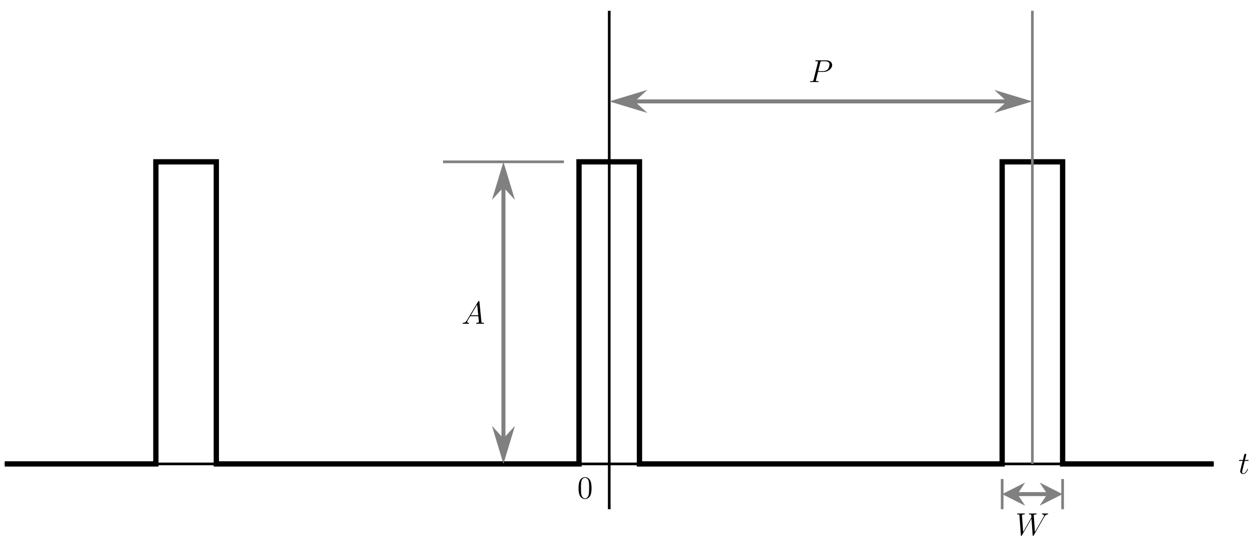
Let R be the number of samples per second. (e.g., 44100). Let $S \subset \mathbb{R}$ be defined by

$$S = \{x \in \mathbb{R} : \frac{mn - \frac{1}{2}}{R} \leq x \leq \frac{mn + \frac{1}{2}}{R} \text{ for some } n \in \mathbb{Z}\}.$$

Then the waveform f generated by the sequence a_n is a constant times the characteristic function of S, say

$$f(t) = A\chi_S(t).$$

The waveform looks like this, continuing periodically in both the positive and negative directions:



For our standard audio file generating method, we have $A = \frac{20000}{32768}$ and $W = \frac{1}{44100}$ seconds. The value of P, the period, depends on the sequence: $P = \frac{m}{44100}$.

We can find the fourier series of f:

$$f(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos k\omega_1 t + B_k \sin k\omega_1 t$$

where $\omega_1 = \frac{2\pi}{P}$ and the fourier coefficients are given by

$$A_0 = \frac{2}{P} \int_0^P f(t)\, dt$$

$$A_k = \frac{2}{P} \int_0^P f(t) \cos k\omega_1 t\, dt$$

$$B_k = \frac{2}{P} \int_0^P f(t) \sin k\omega_1 t\, dt.$$

These integrals may be evaluated over any interval of length P, and it is easier in our case to use $-\frac{P}{2} \leq t \leq \frac{P}{2}$.
Thus we find:

$$
\begin{aligned}
A_0 &= \frac{2}{P}\int_{-P/2}^{P/2} f(t)\,dt = \frac{2AW}{P} = \frac{2A}{m} \\
A_k &= \frac{2}{P}\int_{-P/2}^{P/2} f(t)\cos k\omega_1 t\,dt = \frac{4}{P}\int_{0}^{W/2} f(t)\cos k\omega_1 t\,dt = \frac{4A}{P}\frac{\sin(k\omega_1\frac{P}{2})}{k\omega_1} = \frac{2A}{\pi k}\sin\left(\pi k\frac{W}{P}\right) \\
B_k &= \frac{2}{P}\int_{-P/2}^{P/2} f(t)\sin k\omega_1 t\,dt = 0
\end{aligned}
$$

Hence, we have, more simply

$$f(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k\frac{44100}{m}t\right)$$

where

$$A_0 = \frac{2A}{m},\ A_k = \frac{2A}{k\pi}\sin\left(\frac{k\pi}{m}\right) \text{ for } k \geq 1.$$

The function $\cos(at)$ has frequency $\frac{a}{2\pi}$, so the fundamental frequency of the function f above is $\frac{44100}{m}$: the fourier series is a sum of cosine waves, one for every positive multiple of this frequency.

Since 44100/2 = 22050 is above the range of hearing (for most people), we actually only have to consider at most $m/2$ harmonics: adding additional harmonics will not make any audible difference in the waveform.

Since $\frac{1}{k}\sin(\frac{\pi k}{m})$ is approximately equal to $\frac{\pi}{m}$ when $k = 1$ and it equals $\frac{2}{m}$ when $k = \frac{m}{2}$, the amplitudes of these harmonics are fairly similar, the largest being around 60 percent larger than the smallest. This results in spectrograms with fairly similarly weighted spectral lines.