Difference sequences of Beatty sequences

Let α be a real number, $\alpha > 1$. Suppose α is not an integer.

We consider the set of differences of the Beatty sequence $\{\lfloor \alpha n \rfloor\}_n$.

Note that, for any real x, we have

$$x - 1 < \lfloor x \rfloor \leq x.$$

Then

$$\alpha(n+1) - 1 < \lfloor \alpha(n+1) \rfloor \leq \alpha(n+1)$$

and

$$\alpha n - 1 < \lfloor \alpha n \rfloor \leq \alpha n,$$

and hence

$$\alpha(n+1) - 1 - \alpha(n) < \lfloor \alpha(n+1) \rfloor - \lfloor \alpha(n) \rfloor < \alpha(n+1) - (\alpha n - 1),$$

that is,

$$\alpha - 1 < \lfloor \alpha(n+1) \rfloor - \lfloor \alpha(n) \rfloor < \alpha + 1.$$

Since $\lfloor \alpha(n+1) \rfloor - \lfloor \alpha(n) \rfloor$ is an integer, and α is not, we may conclude that

$$\lfloor \alpha(n+1) \rfloor - \lfloor \alpha(n) \rfloor = \lfloor \alpha \rfloor \text{ or } \lfloor \alpha \rfloor + 1.$$

Thus, in the difference sequence of a Beatty sequence, there cannot be more than two distinct values. For example, the difference sequence for the Beatty sequence with $\alpha = \sqrt{2}$ begins

$$1, 2, 1, 2, 1, 1, 2, 1, 2, 1, 1, 2, 1, 2, 1, 2, 1, 1, 2, 1, 2, 1, 1, 2, ...$$

and hence the entire sequence consists only of ones and twos.

Further, with $\{x\}$ denoting the fractional part of x (so that $x = \lfloor x \rfloor - \{x\}$), we have

$$\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor = \alpha + \{\alpha n\} - \{\alpha n + \alpha\}.$$

We note

$$\{x+y\} = \begin{cases} \{x\} + \{y\} & \text{if } \{x\} + \{y\} < 1 \\ \{x\} + \{y\} - 1 & \text{if } \{x\} + \{y\} > 1. \end{cases}$$

By the uniform distribution modulo 1 of the sequence $\{m\gamma\}_m$ for any irrational γ, there exist infinitely many integers $\widehat{n}$ such that

$$\{\alpha \widehat{n}\} + \{\alpha\} > 1$$

and hence

$$\lfloor \alpha(\widehat{n}+1) \rfloor - \lfloor \alpha \widehat{n} \rfloor = \alpha + \{\alpha \widehat{n}\} - \{\alpha \widehat{n} + \alpha\} = \alpha + \{\alpha \widehat{n}\} - (\{\alpha \widehat{n}\} + \{\alpha\} - 1) = \lfloor \alpha \rfloor + 1.$$

Hence, $\lfloor \alpha \rfloor + 1$ will appear infinitely many times in the difference sequence of the Beatty sequence based on α.

By a similar argument, we can conclude as well that $\lfloor \alpha \rfloor$ will also appear infinitely many times in the difference sequence.