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You get three turns to try to achieve certain sets of dice. After the first and second turns, you can keep any of the thrown dice and re-roll the others.

Suppose our goal is to get all the faces to be the same (this is called Yahtzee) and we procede in the most effective manner.

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Suppose those come up 155.

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If the two came up 11, we would have achieved Yahtzee.

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These calculations are summarized in the following transition matrix:

$$P = \begin{pmatrix} 0 & \frac{120}{1296} & \frac{900}{1296} & \frac{250}{1296} & \frac{25}{1296} & \frac{1}{1296} \\ 0 & \frac{120}{1296} & \frac{900}{1296} & \frac{250}{1296} & \frac{25}{1296} & \frac{1}{1296} \\ 0 & 0 & \frac{120}{216} & \frac{80}{216} & \frac{15}{216} & \frac{1}{216} \\ 0 & 0 & 0 & \frac{25}{36} & \frac{10}{36} & \frac{1}{36} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We start in the zero state and wish to achieve state 5 in at most three rolls.

Looking at the third power of P, we find the probability that we are looking for as the rightmost entry in the first row: this is the probability of transitioning from the 0 state to the 5 state in three or fewer rolls:

$$P^3 = \begin{pmatrix} 0 & \frac{125}{157464} & \frac{26875}{104976} & \frac{3419375}{7558272} & \frac{115625}{472392} & \frac{347897}{7558272} \\ 0 & \frac{125}{157464} & \frac{26875}{104976} & \frac{3419375}{7558272} & \frac{115625}{472392} & \frac{347897}{7558272} \\ 0 & 0 & \frac{125}{729} & \frac{7625}{17496} & \frac{7375}{23328} & \frac{5359}{69984} \\ 0 & 0 & 0 & \frac{15625}{46656} & \frac{11375}{23328} & \frac{8281}{46656} \\ 0 & 0 & 0 & 0 & \frac{125}{216} & \frac{91}{216} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

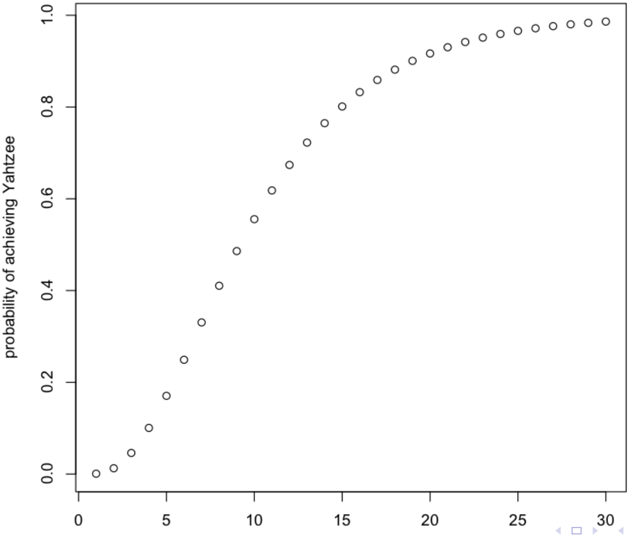
Hence, the probability of achieving Yahtzee is

$$\frac{347897}{7558272} = \frac{1}{22} + \frac{1}{1742} + \frac{1}{22594635} + \cdots \approx 0.0460286425$$

so we expect success about once every 22 attempts.

In general, we can find the probability of achieving Yahtzee in m rolls by looking at the right-most entry in the first row of P^m.

Here is a plot of the probability of achieving Yahtzee with from 1 to 30 rolls.



Probabilistic model of natural language text

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Then we can consider every two-letter sequence (bigram) and count how many times each appears (e.g., how many times an 'a' is followed by an 'a', by a 'b', etc.).

From this we can empirically create a matrix P where, for i and j letters of the alphabet, P_{ij} is the probability that letter i is followed by letter j.

That is, we can treat the text as a Markov chain, with each letter representing a different state, and we move from state to state as we move through the text.

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One can also apply the same idea to words, rather than letters. (One difficulty is the much reduced sample size.)