Some useful notation for chess-related IP designs

It can be challenging to succinctly write the constraints needed in an IP to, for example, find the minimum number of knights needed to attack all squares on a board. Here is a way to do that.

Suppose our board is $m \times n$, with squares indexed from 0 to $m-1$ and from 0 to $n-1$. That is, each square on the board can be represented by an ordered pair (i, j) with $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.

For each square (i, j), we need at least one knight to attack it.

Let S_{ij} be the set of locations from which a knight could attack (i, j); for now, let's ignore the board edges and allow locations that are not actually on the board.

Then we can fully specify S_{ij} by

$$S_{ij} = \{(i \pm 2, j \pm 1)\} \cup \{(i \pm 1, j \pm 2)\}$$

as determined by the rules of knight movement.

Now, some of these locations may not be on the board.

For example, if $(i, j) = (0, 0)$, then $(-2, 1) \in S_{ij}$, but $(-2, 1)$ is not on the board.

So, let's define a new set, A_{ij} of *actual* locations on the board from which a knight can attack (i, j). We can do this by intersecting S_{ij} with the set of all locations on the board:

$$A_{ij} = S_{ij} \cap \{(i, j) : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}.$$

And that's it: in order for (i, j) to be attacked, there must be a knight at at least one location in A_{ij}.

Let's introduce a variable K_{ij} that indicates whether or not there is a knight at (i, j): $K_{ij} = 1$ if there is a knight at (i, j) and otherwise $K_{ij} = 0$.

Then, in our IP, we can require that location (i, j) is attacked by at least one knight with the constraints

$$\sum_{(a,b) \in A_{ij}} K_{ab} \geq 1 \text{ for all } 0 \leq i \leq m-1, 0 \leq j \leq n-1.$$

The summation expression says we will sum up K_{ab} for all (a, b) in A_{ij}, and that is precisely what we want to do: if the sum is at least one, at least one of these K_{ab} variables must be equal to one.